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The asymptotic Z-transform

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THE ASYMPTOTIC Z-TRANSFORM

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This work is dedicated to my grandfathers who passed away in 2004.

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ABSTRACT

Sequences of numbers and transformations from sequences to functions have been studied extensively, including the multiplication of two sequences through convolution and the equivalent multiplication of functions. The focal points of this thesis are the convolution field of causal sequences, $a = (\ldots, 0, 0, a_k, a_{k+1}, a_{k+2}, \ldots)$ with $k \in \mathbb{Z}$, and their Z-transforms $Z(a) := \sum a_i z^i$. Classically, the treatment of the Z-transform has been limited to those causal sequences for which the power series $\sum a_i z^i$ has a nontrivial radius of convergence. In this thesis it is shown that the Z-transform can be extended to all causal sequences without compromising any of the operational properties of the classical Z-transform.

1 CONVOLUTION

1.1 The Field of Causal Sequences

In signal processing, the Z-transform converts a time domain signal, which is a sequence of real numbers $a = (a_0, a_1, a_2, \ldots)$ into a complex frequency domain representation $Z(a)$. Depending on the reference one uses, $Z(a)$ is either defined to be the power series

$$Z(a) = a_0 + a_1 z + a_2 z^2 + \ldots,$$

or a Laurent series

$$Z(a) = a_0 + \tfrac{a_1}{z} + \tfrac{a_2}{z^2} + \ldots \quad .$$

Clearly, the two representations are equivalent up to the substitution of z by $\frac{1}{z}$ and vice versa. In the mathematical literature, especially in combinatorics, the Z-transform is also known as the method of generating functions [13]. The history of the Z-transform seems to be unclear. According to Kirk and Strum [5], the Z-transform was used first by Gardner and Barnes in the early 1940's to solve linear, constant-coefficient difference equations and by W. Hurewicz in 1947 to transform a sample signal or sequence. According to a website maintained by Professor Mark Liberman of the University of Pennsylvania, the term Z-transform originated in 1952 from a sampled-data control group at Columbia University led by Professor John R. Raggazini [6].

Our approach to the Z-transform is somewhat different from the literature in that we consider causal sequences $a = (\ldots, 0, 0, a_k, a_{k+1}, a_{k+2}, \ldots)$, where $k \in \mathbb{Z}$. Such sequences "causal" since they start at some time k (i.e., something caused the observations a_j to start at time $j = k$). In Chapter 2 we study the Z-transform

$$Z(a) := a_k z^k + a_{k+1} z^{k+1} + a_{k+2} z^{k+2} + \ldots,$$

assuming that the power series converges; i.e., we will assume that

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} < \infty. \tag{1}$$

In Chapter 3 this assumption will be removed. For arbitrary sequences

$$a = (\ldots, 0, 0, a_k, a_{k+1}, a_{k+2}, \ldots),$$

the power series $Z(a) := a_k z^k + a_{k+1} z^{k+1} + a_{k+2} z^{k+2} + \ldots$ might no longer converge for any $z \neq 0$. Therefore, we adopt an asymptotic viewpoint and define the asymptotic Z-transform of a time signal $a = (\ldots, 0, 0, a_k, a_{k+1}, a_{k+2}, \ldots)$ as the equivalence class of all functions f which

1. are defined on some sectorial region $S_r^\theta = \{\lambda \in \mathbb{C} \,:\, |\arg \lambda| \leq \theta,\ 0 < |\lambda| < r\}$, where $0 < r \leq \infty$, $0 \leq \theta \leq \pi$, and

2. have an asymptotic expansion at 0 of the form $\sum a_i z^i$.

Further details of the asymptotic case are found in Chapter 3.

We begin by defining the set of causal sequences and the operation of convolution. A causal sequence is a sequence of the form

$$a := (a_0, (a_{-1}, a_1), (a_{-2}, a_2), \ldots)$$

but one usually writes this as

$$a := (\ldots, 0, 0, a_k, a_{k+1}, a_{k+2}, \ldots), \tag{2}$$

where $k \in \mathbb{Z}$, and each element of the sequence is a complex (or real) number. If $a_k \neq 0$ and $a_j = 0$ for all $j < k$, then k is called the lower index of a and write

$$k_l = ind_l(a) := k.$$

If, in addition, $a_{k_u} \neq 0$ and $a_k = 0$ for all $k > k_u$ then the sequence is said to be finite, k_u is called the upper index of a and we write

$$ind_u(a) := k_u.$$

If, for all $k \in \mathbb{N}$, there exists $k_1 > k$ such that $a_{k_1} \neq 0$, we say that $ind_u(a) = \infty$.

Denote by m the set of all causal sequences. Clearly, m becomes a vector space with coordinate-wise addition and scalar multiplication. Define a multiplication, $*$, on m by the convolution (Cauchy product) $a * b = c$, where

$$c_k := \sum_{i=-\infty}^{\infty} a_{k-i} b_i = \sum_{i+j=k} a_i b_j. \tag{3}$$

Notice that the sum in the definition of c_k is finite since $b_i = 0$ for i less than the lower index of b and $a_{k-i} = 0$ for i larger than $k - ind_l(a)$, i.e.,

$$c_k = \sum_{i=ind_l(b)}^{k-ind_l(a)} a_{k-i} b_i. \tag{4}$$

Denote the convolution product of more than one sequence by

$$\prod_{i=1}^{n} a_n := a_1 * a_2 * \ldots * a_n, \quad n \in \{\mathbb{N} \cup \infty\}, \quad \text{and write } a^n \text{ for } \underbrace{a * a * a * \ldots * a}_{n \text{ times}}.$$

Proposition 1.1. *For $a, b \in m$,*

$$ind_l(a * b) = ind_l(a) + ind_l(b). \tag{5}$$

Moreover, if $a, b \in m$ are finite sequences, that is $ind_u(a) < \infty$ and $ind_u(b) < \infty$, then

$$ind_u(a * b) = ind_u(a) + ind_u(b). \tag{6}$$

Proof. For part 1, if $k - ind_l(a) < ind_l(b)$, or equivalently, if $k < ind_l(a) + ind_l(b)$, then $c_k = 0$. Thus, $ind_l(a * b) \geq ind_l(a) + ind_l(b)$. If $k = ind_l(a) + ind_l(b)$, then $c_k = a_{ind_l(a)} b_{ind_l(b)} \neq 0$. For part 2, let a be a causal sequence with $ind_l(a) = k_a$ and $ind_u(a) = k_a + n$ for some $n \in \mathbb{N}$. Let b be a causal sequence with $ind_l(b) = k_b$ and $ind_u(b) = k_b + m$, for some $m \in \mathbb{N}$. Without loss of generality, we can assume that $k_a + n \leq k_b + m$. Then $c_k = \sum\limits_{i=-\infty}^{\infty} a_{k-i} b_i$. Since $a_{k-i} = 0$ for $k - i > k_a + n$ or equivalently $a < k - k_a - n$, it follows that $c_k = \sum\limits_{i=k-k_a+n}^{\infty} a_{k-i} b_i$. Also, since $b_i = 0$ for $i > k_b + m$, then $c_k = \sum\limits_{i=k-k_a+n}^{k_b+m} a_{k-i} b_i$. In particular, $c_k = 0$ when $k - k_a - n > k_b + m$ or equivalently $k > k_b + m + k_a + n$. When $k = k_b + m + k_a + n$, then $c_k = \sum\limits_{i=k_b+m}^{k_b+m} a_{k_a+n} b_i = a_{k_a+n} b_{k_b+m} \neq 0$. Therefore, $ind_u(a * b) = k_a + n + k_b + m = ind_u(a) + ind_u(b)$. ∎

Proposition 1.2. *Let $a = (\ldots, 0, 0, a_{k_0}, a_{k_0+1}, \ldots)$ and $b = (\ldots, 0, 0, b_{k_1}, b_{k_1+1}, \ldots)$, then*

$$a * b = (\ldots, 0, 0,\ a_{k_0} b_{k_1},\ a_{k_0+1} b_{k_1} + a_{k_0} b_{k_1+1},\ a_{k_0+2} b_{k_1} + a_{k_0+1} b_{k_1+1} + a_{k_0} b_{k_1+2}, \ldots), \quad (7)$$

where $a_{k_0} b_{k_1}$ is in the $(k_0 + k_1)$-th position. Moreover, if $a * b = 0$, then either $a = 0$ or $b = 0$.

Proof. Equation (7) follows directly from the definition of the convolution product. If $a \neq 0$ and $b \neq 0$, then $a_{k_0} \neq 0$ and $b_{k_1} \neq 0$. Thus, by Equation (7), $c_{k_0+k_1} = a_{k_0} b_{k_1} \neq 0$, where $c = (c_n)_{n \in \mathbb{Z}} = a * b$. Therefore, $a * b \neq 0$. ∎

Example 1.3. Let $a = (\ldots, 0, \underset{\underset{a_{-2}}{\uparrow}}{1}, 1, 1, \underset{\underset{a_{1}}{\uparrow}}{1}, 0, \ldots)$, and $b = (\ldots, 0, \underset{\underset{b_{1}}{\uparrow}}{1}, 1, 1, 1, 1, \underset{\underset{b_{6}}{\uparrow}}{1}, 0, \ldots)$.

Then $a * b = c$, where $c_k = \sum\limits_{i=-\infty}^{\infty} a_{k-i} b_i$. Since $b_i = 0$ for $i \notin \{1, 2, \ldots, 6\}$ and $b_i = 1$ for $i \in \{1, 2, \ldots, 6\}$, giving

$$c_k = \sum_{i=-\infty}^{6} a_{k-i} b_i = \sum_{i=1}^{6} a_{k-i} = \sum_{j=k-6}^{k-1} a_j.$$

Since $a_j = 0$ for $j \notin \{-2, -1, 0, 1, 2\}$, we obtain $c_k = 0$ if $k - 1 < -2$ or equivalently $k < -1$ and $c_k = 0$ if $k - 6 > 1$ or equivalently $k > 7$.

Moreover, $c_{-1} = \sum\limits_{j=-7}^{-2} a_j = \sum\limits_{j=-2}^{-2} = a_{-2} = 1$, $c_0 = \sum\limits_{j=-6}^{-1} a_j = 2, \ldots,$ $c_7 = \sum\limits_{j=1}^{6} a_j = 1$ and

$$c = (\ldots, 0, 0, \underset{\underset{c_{-1}}{\uparrow}}{1}, 2, 3, 4, 4, 4, 3, 2, \underset{\underset{c_{7}}{\uparrow}}{1}, 0, 0, \ldots).$$

$\blacksquare$

In the following examples, we will see that the convolution product appears naturally when one considers the probability distribution of a sum of independent random variables.

Example 1.4. Consider an experiment in which a fair dice is tossed two times. Denote the probability of rolling the number k on each of the tosses as a_k. The probability distribution of the possible outcomes k, of rolling a fair dice once, can be represented as

$$a = (\ldots, 0, 0, \underset{\substack{\uparrow \\ a_1}}{\tfrac{1}{6}}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \underset{\substack{\uparrow \\ a_6}}{\tfrac{1}{6}}, 0, 0 \ldots),$$

Now consider the random variable consisting of the sum on the faces of the two tosses. The probability of the sum of the tosses being equal to 2, $p\{i+j=2\} = p\{i=1, j=1\} = \frac{1}{36}$. We also have $p\{i+j=3\} = p\{i=1, j=2\} + p\{i=2, j=1\} = \frac{2}{36}$, $p\{i+j=4\} = p\{i=1, j=3\} + p\{i=3, j=1\} + p\{i=2, j=2\} = \frac{3}{36}$, etc. The range of the sum is from 2 to 12, the sample space consists of 36 possibilities, and the probability of each outcome, $p\{i+j=k\}$, can be represented by $a * a$,

$$(a*a)_k = \sum_{i+j=k} a_i a_j = \sum_{i=-\infty}^{\infty} a_{k-i} a_i.$$

Since $ind_l(a*a) = 2$, the first term of the sequence $(a*a)_k$ is $(a*a)_2 = \sum\limits_{i+j=2} a_i a_j = a_1 a_1 = \frac{1}{36}$. The next term is $\sum\limits_{i+j=3} a_i a_j = a_1 a_2 + a_2 a_1 = \frac{2}{36}$. The last term is $(a*a)_{12} = \sum\limits_{i+j=12} a_i a_j = a_6 a_6 = \frac{1}{36}$. So, the convolution operation represents the random variable giving the sum of the two rolls,

$$P\{i+j\} = a*a = (\ldots, 0, 0, \underset{\substack{\uparrow \\ (a*a)_2}}{\tfrac{1}{36}}, \tfrac{2}{36}, \tfrac{3}{36}, \tfrac{4}{36}, \tfrac{5}{36}, \tfrac{6}{36}, \tfrac{5}{36}, \tfrac{4}{36}, \tfrac{3}{36}, \tfrac{2}{36}, \underset{\substack{\uparrow \\ (a*a)_{12}}}{\tfrac{1}{36}}, 0, 0 \ldots).$$

$\blacksquare$

Example 1.5. Consider the sequence $a = (\ldots, 0, 0, \underset{\substack{\uparrow \\ a_{-2}}}{\frac{1}{4}}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, \ldots)$. This can be thought of as a probability distribution representing a spinner which can take on the values $k \in \{-2, -1, 0, 1\}$ with equal probability, and the sequence $b = (\ldots, 0, 0, \underset{\substack{\uparrow \\ b_1}}{\frac{1}{6}}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, \ldots)$ as representing the roll of a fair dice with face value $i \in \{1, 2, 3, 4, 5, 6\}$. Then, the probability a_k of the spinner landing on k is $\frac{1}{4}$ and the probability of the dice showing the value i is always $\frac{1}{6}$. The sum of the spin and the roll, which ranges from -1 to 7, can be represented by the sequence

$$P\{k+i\} = a*b = (\ldots,0,0,\underset{\substack{\uparrow \\ (a*b)_{-1}}}{\tfrac{1}{24}},\tfrac{2}{24},\tfrac{3}{24},\tfrac{4}{24},\tfrac{4}{24},\tfrac{4}{24},\tfrac{3}{24},\tfrac{2}{24},\underset{\substack{\uparrow \\ (a*b)_{7}}}{\tfrac{1}{24}},0,0,\ldots).$$

■

From Examples 1.4 and 1.5, and most importantly from Equation (3), note that the probability distribution of the sum of two independent random variables with probability distributions $a, b \in m$ is given by the convolution product $a * b$. The probability distributions of Examples 1.4 and 1.5 are uniform, as for $a_k \neq 0$, then $a_k = c$, where $0 < c < 1$. The following examples illustrate a non-uniform distribution. Further examples can be found in [4].

Example 1.6. Consider successive tosses of a coin with probability of heads equal to p and probability of tails equal to $1-p$ on each toss. The probability of getting k heads in n successive tosses is given by the sequence

$$b(n,p) = (\ldots,0,0,0,\underset{\substack{\uparrow \\ P\{k=0\}}}{\tbinom{n}{0}(1-p)^n},\tbinom{n}{1}p(1-p)^{n-1},\tbinom{n}{2}p^2(1-p)^{n-2},\ldots,\underset{\substack{\uparrow \\ P\{k=n\}}}{\tbinom{n}{n}p^n},0,0,0,\ldots).$$

Intuitively, we can predict that the result would be the same if a single coin were tossed n times or n identical coins were tossed each once, i.e.,

$$b(n,p) = b(1,p)^n,$$

where $b(1,p) = (\ldots,0,0,\underset{\substack{\uparrow \\ P\{k=0\}}}{1-p},p,0,0,\ldots)$. Algebraically, this can be easily verified using induction. More generally,

$$b(n,p) * b(m,p) = b(1,p)^n * b(1,p)^m = b(1,p)^{n+m} = b(n+m,p).$$

■

Example 1.7. In many applications we encounter trials in which n is so large that the product $\lambda = np$ is of substantial magnitude. In this case,

$$b(n,p)_k = \tbinom{n}{k}(1-p)^{n-k}p^k = \tfrac{n!}{k!(n-k)!}(1-\tfrac{\lambda}{n})^{n-k}(\tfrac{\lambda}{n})^k =$$

$$\tfrac{n(n-1)\ldots(n-k+1))}{n \cdot n \ldots n}\tfrac{\lambda^n}{k!}\tfrac{1}{(1-\frac{\lambda}{n})^k}(1-\tfrac{\lambda}{n})^n \approx \tfrac{\lambda^k}{k!}e^{-\lambda}$$

if λ is fixed and n large. Thus, if $\lambda = np$ when n is large, then a convenient approximation to the binomial distribution, $b(n,p)$, is the Poisson distribution given by the sequence

$$p(\lambda) = (\tfrac{e^{-\lambda}\lambda^k}{k!})_{k\geq 0} = e^{-\lambda}(\ldots,0,0,0,\underset{\substack{\uparrow \\ P\{k=0\}}}{1},\lambda,\tfrac{\lambda^2}{2!},\tfrac{\lambda^3}{3!},\tfrac{\lambda^4}{4!},\tfrac{\lambda^5}{5!},\ldots).$$

For Poisson distributions, we have that

$$p(\lambda) * p(\mu) = p(\lambda + \mu),$$

since $\sum_{i=0}^{k} p(\lambda)_i p(\mu)_{k-i} = \sum_{i=0}^{k} e^{-\lambda}\frac{\lambda^i}{i!} e^{-\mu}\frac{\mu^{k-i}}{(k-i)!} = e^{-(\lambda+\mu)}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\lambda^i\mu^{k-i} = e^{-(\lambda+\mu)}\frac{1}{k!}(\lambda+\mu)^k.$

$\blacksquare$

In the examples above we dealt with probability distributions $a = (a_k)_{k\in\mathbb{Z}}$. In these cases, $a_k \geq 0$ for all k and $\sum_{k=-\infty}^{\infty} a_k = 1$. We have seen that if a, b are two probability distributions then $c = a * b$ is a probability distribution with $c_k \geq 0$ for all k and $\sum_{k=-\infty}^{\infty} c_k = 1$. If we define the 1-norm of an absolutely summable sequence $a = (a_k)_{k\in\mathbb{Z}}$ as

$$\|a\|_1 := \sum_{k=-\infty}^{\infty} |a_k|, \tag{8}$$

then these observations can be extended as follows.

Proposition 1.8. *If $a, b \in m$ are absolutely summable sequences, then $a * b$ is absolutely summable and*

$$\|a * b\|_1 \leq \|a\|_1\|b\|_1. \tag{9}$$

Moreover, if $a_k, b_k \geq 0$, then $\|a * b\|_1 = \|a\|_1\|b\|_1$.

Proof. Clearly, $\|a * b\|_1 = \sum_{k=-\infty}^{\infty} |\sum_{i+j=k} a_i b_j| \leq \sum_{k=-\infty}^{\infty}\sum_{i+j=k} |a_i||b_j| = \|a\|_1\|b\|_1$ (see also Equation (18) which provides a more detailed argument). Now let a and b be sequences with $a_k, b_k \geq 0$ for all k. Define two positive sequences $\tilde{a}$ and $\tilde{b}$ by $\tilde{a} := \frac{a}{\|a\|_1}$, and $\tilde{b} := \frac{b}{\|b\|_1}$. Then $\|\tilde{a}\|_1 = 1$ and $\left\|\tilde{b}\right\|_1 = 1$. From Equations (4) and (8), it follows that

$$\begin{aligned}\|\tilde{a} * \tilde{b}\|_1 &= \sum_{k=-\infty}^{\infty}(\sum_{i+j=k}\tilde{a}_i\tilde{b}_j) = \sum_{i=-\infty}^{\infty}\tilde{a}_i\sum_{k=-\infty}^{\infty}\tilde{b}_k = \sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\tilde{a}_{k-i}\tilde{b}_i \\ &= \sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\tilde{a}_{k-i}\tilde{b}_i = \sum_{i=-\infty}^{\infty}\tilde{b}_i\sum_{k=-\infty}^{\infty}\tilde{a}_{k-i} = 1\end{aligned}$$

Therefore, $\|\tilde{a} * \tilde{b}\|_1 = \|\tilde{a}\|_1\|\tilde{b}\|_1$, and $\|a * b\|_1 = \|a\|_1\|b\|_1$. $\blacksquare$

We denote by $e_k := (\ldots, 0, 0, \underset{\substack{\uparrow \\ k\text{-th position}}}{1}, 0, 0, \ldots)$ the k-th unit sequence, and in particular

$$e_0 := (\ldots, 0, 0, 0, \underset{\substack{\uparrow \\ 0\text{-th position}}}{1}, 0, 0, 0, \ldots).$$

It is sometimes convenient to write a sequence $a = (a_k)_{k\in\mathbb{Z}}$ in the form $a = \sum_{k=-\infty}^{\infty} a_k e_k$. It is important to note that this is a purely formal "infinite" sum; that is, there is no limiting procedure involved. The sole meaning of $\sum_{k=-\infty}^{\infty} a_k e_k$ is that it coincides with $a = (a_k)_{k\in\mathbb{Z}}$.

Theorem 1.9. *The set of causal sequences, m, along with the operations of addition and convolution, $(m, +, *)$, is a field with multiplicative identity $I := e_0$.*

Proof. It is clear that $(m, +)$ forms an abelian group. We now show that the convolution operation is commutative and associative, that e_0 is the multiplicative identity, and that the distributive law holds in $(m, +, *)$. Commutativity follows from

$$(a*b)_k = \sum_{i=-\infty}^{\infty} a_{k-i}b_i \underset{\substack{\uparrow \\ j=k-i}}{=} \sum_{j=-\infty}^{\infty} a_j b_{k-j} = \sum_{j=-\infty}^{\infty} b_{k-j}a_j = (b*a)_k.$$

For associativity of the convolution operation, observe first that

$$((a*b)_i * c)_k = \sum_{i=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{\infty} a_{i-j}b_j\Big)c_{k-i} = \Big(\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} a_{i-j}b_jc_{k-i}\Big)_k = \Big(\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} a_{i-j}b_jc_{k-i}\Big)_k$$

$$= \Big(\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} a_{i-j}b_jc_{k-i}\Big)_k = \Big(\sum_{j=-\infty}^{\infty} b_j \sum_{i=-\infty}^{\infty} a_{i-j}c_{k-i}\Big)_k.$$

Let $m = k - i + j$, then $i = k - m + j$, and

$$\Big(\sum_{j=-\infty}^{\infty} b_j \sum_{i=-\infty}^{\infty} a_{i-j}c_{k-i}\Big)_k = \Big(\sum_{j=-\infty}^{\infty} b_j \sum_{m=-\infty}^{\infty} a_{k-m}c_{m-j}\Big)_k.$$

Let $m = i$, then

$$\Big(\sum_{j=-\infty}^{\infty} b_j \sum_{m=-\infty}^{\infty} a_{k-m}c_{m-j}\Big)_k = \Big(\sum_{j=-\infty}^{\infty} b_j \sum_{i=-\infty}^{\infty} a_{k-i}c_{i-j}\Big)_k = \Big(\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty} a_{k-i}b_jc_{i-j}\Big)_k$$

$$= \Big(\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} a_{k-i}b_jc_{i-j}\Big)_k = \Big(\sum_{i=-\infty}^{\infty} a_{k-i}\sum_{j=-\infty}^{\infty} b_jc_{i-j}\Big)_k = (a*(c*b))_k$$

$$= (a*(b*c))_k \text{ by commutativity.}$$

That e_0 is the multiplicative identity follows from

$$(a*e_0)_k = \sum_{i+j=k} a_i(e_0)_j = a_k, \text{ since } (e_0)_j = 1 \text{ if } j = 0 \text{ and } (e_0)_j = 0 \text{ if } j \neq 0.$$

By commutativity, $(a * e_0)_k = (e_0 * a)_k = a_k$. The distributivity of the convolution operation follows from

$$(a*(b+c))_k = \sum_{i=-\infty}^{\infty} a_{k-i}(b+c)_i = \sum_{i=-\infty}^{\infty} a_{k-i}(b_i+c_i)$$

$$= \sum_{i=-\infty}^{\infty} a_{k-i}b_i + a_{k-i}c_i = (a * b + a * c)_k.$$

Having established $(m, +, *)$ as a ring, we further prove that for all $a \in m, a \neq 0$, there exists an unique $b \in m$ such that $a * b = e_0$. To show uniqueness, recall that by Equation (7) that $(m, +, *)$ is an integral domain; i.e., if $a * b = 0$, then either $a = 0$ or $b = 0$. Now assume there exists $b_1, b_2 \in m$ such that $a * b_1 = e_0$ and $a * b_2 = e_0$ for $a \neq 0$. Then $a*(b_1 - b_2) = 0$. Thus, $b_1 - b_2 = 0$ or $b_1 = b_2$. To show existence of a multiplicative inverse, let a be a causal sequence with $ind_l(a) = k$. Then, by Equation (5), we can assume that the inverse has the form

$$b = (\ldots, 0, 0, b_{-k}, b_{-k+1}, b_{-k+2}, \ldots).$$

Then $e_0 = a * b = (\ldots, 0, 0,\ \ a_k b_{-k},\ \ a_{k+1}b_{-k} + a_k b_{-k+1},\ \ a_{k+2}b_{-k} + a_{k+1}b_{-k+1} + a_k b_{-k+2},$

$$a_{k+3}b_{-k} + a_{k+2}b_{-k+1} + a_{k+1}b_{-k+2} + a_k b_{-k+3}, \ldots)$$

Equating terms and solving for the coordinates of $b = a^{-1}$, we have

$$b_{-k} = \tfrac{1}{a_k},$$

$$b_{-k+1} = -\tfrac{1}{a_k}(a_{k+1}b_{-k}) = -\tfrac{a_{k+1}}{a_k^2},$$

$$b_{-k+2} = -\tfrac{1}{a_k}(a_{k+2}b_{-k} + a_{k+1}b_{-k+1}) = -\tfrac{1}{a_k}(\tfrac{a_{k+2}}{a_k} - \tfrac{a_{k+1}^2}{a_k^2}),$$

$b_{-k+3} = -\tfrac{1}{a_k}(a_{k+3}b_{-k} + a_{k+2}b_{-k+1} + a_{k+1}b_{-k+2})$, and in general for $m \in \mathbb{N}$,

$$b_{-k+m} = -b_{-k} \sum_{i=1}^{m} a_{k-i+m+1} b_{-k+i-1}, \tag{10}$$

A Mathematica program to find the first hundred elements of the inverse of a finite length causal sequence is given in the Appendix. ■

Example 1.10. Consider the sequence $a = (\ldots, 0, 0, 0, \underset{\substack{\uparrow \\ \text{0-th position}}}{1}, 2, 3, 4, 5, 6, \ldots).$

Then, by Equation (10), $a^{-1} = b$, where

$$b_0 = \tfrac{1}{a_0} = 1,$$

$$b_1 = -a_1 b_0 = -2,$$

$b_2 = -(a_2 b_0 + a_1 b_1) = -(3 - 4) = 1$, and

$$b_3 = -(a_3 b_0 + a_2 b_1 + a_1 b_2) = -(4 - 6 + 2) = 0.$$

By induction that $b_n = 0$ for $n \geq 3$. Suppose the statement is true for some $n \geq 3$. Then $b_{n+1} = -(a_{n+1}b_0 + a_n b_1 + a_{n-1}b_2) = -((n+2) - 2(n+1) + n) = 0$. Thus,

$$a^{-1} = (\ldots, 0, 0, \underset{\substack{\uparrow \\ \text{0-th position}}}{1}, -2, 1, 0, 0, \ldots).$$

■

Example 1.11. Let $a = (\ldots, 0, \underset{\substack{\uparrow \\ \text{0-th position}}}{1}, 1, 1, 1, 0, \ldots)$, which is the same sequence a of Example 1.3. From the program in the Appendix that computes the first n terms of the inverse a^{-1}, we suggest that

$$a^{-1} = b = (\ldots, 0, 0, \underset{\substack{\uparrow \\ \text{0-th position}}}{1}, -1, 0, 0, 1, -1, 0, 0, 1, -1, 0 \ldots),$$

with a repeating pattern of $(1, -1, 0, 0)$. To prove our hypothesis we compute $a * b$ and find that $ind_l(a * b) = 0$, $(a * b)_0 = 1$,

$$(a * b)_1 = a_0 b_1 + a_1 b_0 = b_1 + b_0 = 0,$$

$$(a * b)_2 = a_2 b_0 + a_1 b_1 + a_0 b_2 = b_0 + b_1 = 0, \text{ and}$$

$$(a * b)_3 = a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3 = 0.$$

For $n \geq 4$, $(a * b)_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + a_3 b_{n-3} = b_n + b_{n-1} + b_{n-2} + b_{n-3} = 0$ since the sum of any 4 consecutive coordinates of b equals 0. This shows that $a * b = e_0$ or $b = a^{-1}$.

■

Example 1.12. Let $a = (\ldots, 0, 0, 1, 1, \frac{1}{2}, \frac{1}{3!}, \frac{1}{4!}, \ldots)$, where the first term is in the 0-th position. Then $a^{-1} = b$, where $b_0 = 1$, $b_1 = -1$,

$$b_2 = -(a_2 b_0 + a_1 b_1) = -(\tfrac{1}{2} - 1) = \tfrac{1}{2}, \text{ and}$$

$$b_3 = -(a_3 b_0 + a_2 b_1 + a_1 b_2) = -(\tfrac{1}{3!} - \tfrac{1}{2!} + \tfrac{1}{2!}) = -\tfrac{1}{3!}.$$

This suggests that $b_n = (-1)^n \frac{1}{n!}$. Suppose the statement is true for all $0 \leq i \leq n$. Then

$$b_{n+1} = -(a_{n+1} b_0 + a_n b_1 + a_{n-1} b_2 + \ldots + a_1 b_n)$$

$$= -(\tfrac{1}{(n+1)!} - \tfrac{1}{n!} + \tfrac{1}{2!(n-1)!} - \tfrac{1}{3!(n-2)!} + \ldots + (-1)^n \tfrac{1}{n!})$$

$$= (-1)^{n+1} \tfrac{1}{(n+1)!} [(-1)^n + (-1)^{n-1} \tfrac{(n+1)!}{n!} + (-1)^{n-2} \tfrac{(n+1)!}{2!(n-1)!} + \ldots + (-1)^0 \tfrac{(n+1)!}{n!}]$$

$$= (-1)^{n+1} \tfrac{1}{(n+1)!} \sum_{k=0}^{n} (-1)^{n-k} \tbinom{n+1}{k} = (-1)^{n+1} \tfrac{1}{(n+1)!} [- \sum_{k=0}^{n} (-1)^{n+1-k} \tbinom{n+1}{k}]$$

$$= (-1)^{n+1}\tfrac{1}{(n+1)!}[-\sum_{k=0}^{n+1}(-1)^{n+1-k}\tbinom{n+1}{k} + 1]$$

$$= (-1)^{n+1}\tfrac{1}{(n+1)!}[-(1-1)^{n+1} + 1] = (-1)^{n+1}\tfrac{1}{(n+1)!}.$$

∎

The examples above show that for infinite sequences $0 \neq a \in m$, the inverse may be finite or infinite. In Proposition 2.12 we will characterize those infinite sequences which have a finite inverse using the Z-transform. If a is finite (i.e., $ind_u(a) \leq \infty$), then we distinguish between two cases. If a has exactly one nontrivial coordinate a_k; i.e., $a = (\ldots, 0, 0, a_k, 0, 0, \ldots)$, then

$$a^{-1} = b = (\ldots, 0, 0, b_{-k}, 0, 0, \ldots),$$

where $b_{-k} = \frac{1}{a_k}$. The case where $0 < ind_l(a) < ind_u(a) < \infty$ is addressed in Proposition 1.13. Computing the inverse a^{-1} by the method of the previous two examples is a time-consuming and tedious process. In the next section we will find a more effective way of doing so by means of the Z-transform. We will find that the Z-transform will solve the problem of computing the convolution inverse of a finite sequence by reducing it to these elementary algebra procedures

(1) factoring polynomials (The Fundamental Theorem of Algebra),
(2) partial fraction decomposition, and
(3) the geometric series.

Proposition 1.13. *Let $a \in m$ be a finite sequence with more than one nontrivial coordinate, i.e., $-\infty < ind_l(a) < ind_u(a) < \infty$. Then a^{-1} is infinite; i.e., $ind_u(a^{-1}) = \infty$. Moreover, $ind_l(a^{-1}) = -ind_l(a)$.*

Proof. Assume a^{-1} is finite. Then it follows from Equation (5) and Proposition 1.1 that

$$ind_l(a) + ind_l(a^{-1}) = ind_l(a * a^{-1}) = ind_l(e_0) = 0$$

and

$$ind_u(a) + ind_u(a^{-1}) = ind_u a * a^{-1} = ind_u(e_0) = 0.$$

In particular, $ind_l(a^{-1}) = -ind_u(a)$ and

$$ind_l(a) + ind_l(a^{-1}) = ind_u(a) + ind_u(a^{-1}).$$

The last equality yields $0 < ind_u(a) - ind_l(a) = ind_u(a^{-1}) - ind_l(a^{-1})$. This implies that $ind_u(a^{-1}) < ind_l(a^{-1})$, which is a contradiction. ∎

We would like to end this section with the following problem which was given to us by Professor Hsiao-Chun Wu of the Louisiana State University Electrical Engineering Department, in a slightly different form (using the $\|\cdot\|_2$ norm instead of the $\|\cdot\|_1$ norm).

Problem 1.14. Let $L \in \mathbb{N}$ and m_0^L be the set of finite sequences a with

$$0 = ind_l(a) < ind_u(a) = L < \infty.$$

Also, let $n \in \mathbb{N}$ and $M > 0$ be a large constant. Find $a \in m_0^L$ such that

$$\sum_{i=0}^{n} |a_i^{-1}| \geq M \sum_{n+1}^{\infty} |a_i^{-1}|;$$

i.e., find $a \in m_0^L$ such that "the mass" of a^{-1} is concentrated in the first n coordinates. First of all, if a is a finite sequence then we have seen in the examples above that a^{-1} is of infinite length and that, in general, a^{-1} is not summable (see Example 1.11). In fact, it is not clear with the tools at hand if such a sequence exists at all, but the problem will be revisited later in Chapter 2 using the Z-transform as the major computational tool.

■

1.2 Shift Invariant Operators

In Section 1.1 the convolution product appeared naturally in probability theory for probability distributions of sums of independent random variables. Another application in which the convolution product appears naturally is in signal processing (filter design). Let $a \in m$ be a finite sequence representing a time signal; i.e., $-\infty < ind_l(a) \leq ind_u(a) < \infty$, and let

$$m_0 = \{a \in m\ : a \text{ finite}\}.$$

In designing a signal processor (filter) $F\ :\ m_0 \to m$, it is desirable to obtain linearity (the principle of superposition) and shift-invariance. Shift-invariance is a desirable property since it should not matter if one first processes the time signal $a = (\ldots, 0, 0, a_{k_0}, a_{k_0+1}, a_{k_0+2}, \ldots)$ and then shifts the output by n time units, or if one first shifts the signal a by n time units and then processes the signal

$$(\ldots, 0, 0, a_{k_0+n}, a_{k_0+n+1}, a_{k_0+n+2}, \ldots).$$

We will see in this section that a linear map $F\ :\ m_0 \to m$ is shift invariant if and only if there exists $b \in m$ such that $F(a) = b * a$ for all $a \in m_0$.

Theorem 1.15 (Shift Operators). *Define $T_k\ :\ m \to m$ by $T_k(a) := e_k * a$. Then T_k is linear and*

$$T_k(a) = (a_{n-k})_{n\in\mathbb{N}}; \tag{11}$$

i.e., T_k represents a right shift by k positions. In particular, $e_k = e_1^k$ for all $k \in \mathbb{Z}$, where

$$e_k = (\ldots, 0, 0, \underset{\substack{\uparrow \\ k\text{-th position}}}{1}, 0, 0, \ldots).$$

Proof. T_k is linear since

$$T_k(\lambda_1 a + \lambda_2 b) = e_k * (\lambda_1 a + \lambda_2 b) = \lambda_1(e_k * a) + \lambda_2(e_k * b) = \lambda_1 T_k(a) + \lambda_2 T_k(b)$$

for all $a, b \in m$ with $\lambda_1, \lambda_2 \in \mathbb{C}$. Moreover,

$$T_k(a) = e_k * a = (\sum_{i+j=n} e_{k_i} a_j)_{n\in\mathbb{N}} = (a_{n-k})_{n\in\mathbb{N}}$$

since $e_{k_i} = 0$ if $i \neq k$ and $e_{k_k} = 1$. In particular $T_1(a) = e_1 * a$ is the right shift operator. Since e_k is the sequence e_1 shifted $k-1$ positions to the right, it follows that $e_k = T_1^{k-1}(e_1) = e_1^{k-1} * e_1 = e_1^k$. ■

An immediate application of the fact that the unit-sequences e_k are given by the k-th powers of e_1 is the following extension of the well known algebraic version of geometric series to the convolution field m.

Proposition 1.16 (Geometric Series)**.** *Let $I := e_0$ and $e := e_1$.*

(a) *Let $a \in m$. Then $\sum_{i=0}^{n} a^n = I + a + a^2 \ldots + a^n = (I - a^{n+1})(I - a)^{-1}$.*

(b) $\sum_{i=0}^{\infty} e^i = I+e+e^2+\ldots = e_0+e_1+e_2+\ldots := (\ldots, 0, 0, 1, 1, 1, 1, 1, \ldots) = (I-e)^{-1}$.

(c) *For $a \in m$ of the form $a = (\ldots, 0, 0, 0, \underset{\substack{\uparrow \\ k\text{-th position}}}{\tilde{a}}, \tilde{a}^2, \tilde{a}^3, \tilde{a}^4, \ldots)$ where $\tilde{a} \neq 0 \in \mathbb{C}$,*

then $a^{-1} = (\ldots, 0, 0, 0, \underset{\substack{\uparrow \\ -k\text{-th position}}}{\frac{1}{\tilde{a}}}, -1, 0, 0, 0, \ldots)$.

Proof. (a) Clearly, $(I-a)(I + a + a^2 + \ldots + a^n) = I - a^{n+1}$. Since m is a field we obtain that

$$I + a + a^2 + \ldots + a^n = (I - a^{n+1})(I - a)^{-1}.$$

(b) To show that $\sum_{i=0}^{\infty} e^i = (I-e)^{-1}$, consider the sequence $a = I-e = (\ldots, 0, 0, \underset{\substack{\uparrow \\ \text{0-th position}}}{1}, -1, 0, 0, \ldots)$.

For this sequence and from Equation (10), $a_0^{-1} = 1$, $a_1^{-1} = -\sum_{i=1}^{1} a_{2-i} a_{i-1}^{-1} = 1$, $a_2^{-1} = -\sum_{i=1}^{2} a_{3-i} a_{i-1}^{-1} = 1$, $a_3^{-1} = -\sum_{i=1}^{3} a_{4-i} a_{i-1}^{-1} = 1$, $a_4^{-1} = -\sum_{i=1}^{4} a_{5-i} a_{i-1}^{-1} = 1$, and so on.

Therefore, $a^{-1} = (\ldots, 0, 0, \underset{\substack{\uparrow \\ \text{0-th position}}}{1}, 1, 1, 1, 1, \ldots)$ and $\frac{I}{I-e} = (I-e)^{-1}$

$$= (\ldots, 0, 0, 1, 1, 1, 1, 1, \ldots) = \sum_{i=0}^{\infty} e_i.$$
$$\uparrow$$
0-th position

(c) From Equation (10), $a^{-1}_{-k} = \frac{1}{a_k} = \frac{1}{\tilde{a}}$,

$a^{-1}_{-k+1} = \frac{-1}{\tilde{a}} \sum_{i=1}^{1} a_{k-i+2} a^{-1}_{-k+i-1} = \frac{-1}{\tilde{a}} a_{k+1} a^{-1}_{-k} = -\frac{1}{\tilde{a}} \tilde{a}^2 \frac{1}{\tilde{a}} = -1,$

$a^{-1}_{-k+2} = \frac{-1}{\tilde{a}} \sum_{i=1}^{2} a_{k-i+3} a^{-1}_{-k+i-1} = \frac{-1}{\tilde{a}} (a_{k+2} a^{-1}_{-k} + a_{k+1} a^{-1}_{-k+1}) = \frac{-1}{\tilde{a}} (\tilde{a}^3 \frac{1}{\tilde{a}} + \tilde{a}^2(-1)) = 0,$

$a^{-1}_{-k+3} = \frac{-1}{\tilde{a}} \sum_{i=1}^{3} a_{k-i+4} a^{-1}_{-k+i-1} = \frac{-1}{\tilde{a}} (a_{k+3} a^{-1}_{-k} + a_{k+2} a^{-1}_{-k+1} + a_{k+1} a^{-1}_{-k+2}) = \frac{-1}{\tilde{a}} (\tilde{a}^4 \frac{1}{\tilde{a}} + \tilde{a}^3(-1) + 0) = 0$, and in general, for $m > 3$, $a^{-1}_{-k+m} = \frac{-1}{\tilde{a}} (\tilde{a}^{m+1} \frac{1}{\tilde{a}} + \tilde{a}^m(-1) + 0) = 0.$ ■

Corollary 1.17. *Each shift operator, $T_k : m \to m$, $T_k(a) := e_k * a$, is one-to-one and onto. Its inverse is given by T_{-k}.*

Proof. By definition and from Theorem 1.15, T_k is onto as

$$T_k T_{-k}(a) = T_k(e_{-k} * a) = e_k * (e_{-k} * a) = (e_k * e_{-k}) * a = e_0 * a = a.$$

By a similar argument, we then have $T_{-k} T_k(a) = a$, and thus, T_k is one-to-one. ■

The following remark is useful when computing the inverse of some sequence

$$a = (\ldots, 0, 0, a_k, a_{k+1}, a_{k+2}, \ldots) \in m,$$

where $k = ind_l(a)$. For computational purposes it is often convenient to shift the sequence a by $-k$ units such that the shifted sequence $\tilde{a} := e_{-k} * a = (a_{j+k})_{j \in \mathbb{Z}} =$

$$(\ldots, 0, 0, \underset{\uparrow}{a_k}, a_{k+1}, a_{k+2}, \ldots)$$
0-th position

has lower index equal to zero. Clearly, since $\tilde{a}^{-1} = a^{-1} * e_k$, we get that

$$a^{-1} = e_{-k} * \tilde{a}^{-1}. \tag{12}$$

As mentioned in the introduction to this section, shift invariance is important to applications in signal processing [8], where we can consider a digital signal as a finite sequence that is indexed by units of time. The value of each term in the sequence thus represents some characteristic of the signal. Before continuing on, we give a more precise definition of shift-invariance in terms of the shift operators T_k discussed in Theorem 1.15. An operator $F : m \to m$ is said to be shift-invariant if for all $a \in m$,

$$F(T_k(a)) = T_k(F(a)). \tag{13}$$

Clearly, if $F : m \to m$ is shift-invariant, then so is the restriction $F_0 := F|_{m_0} : m_0 \to m$ of F to $m_0 := \{a \in m : a \text{ finite}\}$. Before characterizing shift-invariant linear operators $F_0 : m_0 \to m$, we have to collect some properties of linear operators.

Write $a \in m_0$ as the finite sum $\sum\limits_n a_n e_n = \sum a_n e^n$, where $e := e^1$, the operator F_0 is completely determined by the action of F_0 on the sequences $e_n = e^n$ (where $e := e_1$) since $F_0(a) = F_0(\sum a_n e^n) = \sum a_n F_0(e_n)$. That is, if $F_0 : m_0 \to m$ is given, then we know the sequences $F_0(e_n) = y^n = (y^n_k)_{k\in\mathbb{Z}} \in m$. Conversely, the action of F on the sequences e_n (i.e., $F_0(e_n) = y^n \in m$), then we know the action of F_0 on all $a \in m_0$ by linearity since

$$F(a) = F(\sum_k a_k e_k) = \sum_k a_k F(e_k) = \sum_k a_k y^n_k = \sum_n a_n \sum_k y^n_k e_k = \sum_k (\sum_n a_n y^n_k) e_k.$$

Theorem 1.18. *Let $F_0 : m_0 \to m$ be linear. The following are equivalent:*

i. *There exists $b \in m$ such that $F_0(a) = a * b$ for all $a \in m_0$.*

ii. *F_0 is shift invariant.*

Proof. Assume that (i) holds; i.e., $F_0(a) = a * b$ for all $a \in m_0$. Then

$$T_k F_0(a) = e_k * (a * b) = (e_k * a) * b = F_0(T_k(a)).$$

Conversely, assume that (ii) holds; i.e., F_0 is shift invariant. Define $b := y^0 := F_0(e_0)$ and $y^n := F_0(e_n) \in m$. Since F is shift invariant and $e_n = e^n$ (where $e = e_1$), it follows that

$$y^n = F_0(e^n) = F_0(T_n(e_0)) = T_n(F_0(e_0)) = e_n * y^0 = e_n * b.$$

Therefore, since $\sum\limits_{n=-\infty}^{\infty} a_n e_n$ is finite ($a \in m_0$!),

$$F_0(a) = F_0(\sum a_n e_n) = \sum a_n T(e_n) = \sum a_n y^n = \sum a_n (e_n * b) = (\sum a_n e_n) * b = a * b.$$

■

Thus, the effect of any linear, shift-invariant operator on an arbitrary finite input sequence is obtained by convolving the input sequence with the response $b := T(e_0)$ of the operator to the unit sequence e_0.

2 THE CLASSICAL Z-TRANSFORM

2.1 Power Series

In this section we will recall some of the basic properties of power series $\sum\limits_{k=0}^{\infty} a_k z^k$ and prove the following fundamental properties (see also [1], [2], [7], and [9]).

Theorem 2.1. *Let* $L := \overline{\lim\limits_{k\to\infty}} \sqrt[k]{|a_k|}$.

(a) *If* $L = 0$*, then* $\sum\limits_{k=0}^{\infty} a_k z^k$ *converges absolutely for all* z.

(b) *If* $L = \infty$*, then* $\sum\limits_{k=0}^{\infty} a_k z^k$ *converges only for* $z = 0$.

(c) *If* $0 < L < \infty$*, then* $R := \frac{1}{L}$ *is the radius of convergence of* $\sum\limits_{k=0}^{\infty} a_k z^k$*. That is,* $\sum\limits_{k=0}^{\infty} a_k z^k$ *converges absolutely for* $|z| < R$ *and diverges for* $|z| > R$*. In particular,* $\sum\limits_{k=0}^{\infty} a_k z^k$ *exists for some* $z \neq 0$ *if and only if* $L < \infty$.

(d) *If* $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ *converges absolutely for* $|z| < R$*, then* $\sum\limits_{k=1}^{\infty} k a_k z^{k-1}$ *converges absolutely for* $|z| < R$*,* f *is differentiable, and* $f'(z) = \sum\limits_{k=1}^{\infty} k a_k z^{k-1}$ *for all* $|z| < R$.

Proof. (a) If $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|} = 0$, then $\lim\limits_{k\to\infty} \sqrt[k]{|a_k|}|z| = 0$ for all $z \in \mathbb{C}$. Thus, for any z there is some N such that $\sqrt[k]{|a_k|}|z| \leq \frac{1}{2}$ for all $k > N$. Therefore, $|a_k z^k| \leq \frac{1}{2^k}$ for all $k > N$. By the comparison test for positive sequences, $\sum\limits_{k=0}^{\infty} a_k z^k$ converges absolutely for all $z \in \mathbb{C}$.

(b) If $L = \infty$, then for any $z \neq 0$, $|a_k|^{\frac{1}{k}} \geq \frac{1}{|z|}$ for infinitely many values of k. Therefore, $|a^k z^k| \geq 1$ for infinitely many k. Thus, the terms of the series do not approach 0, and the series diverges for any $z \neq 0$. The fact that $\sum\limits_{k=0}^{\infty} a_k z^k$ converges always for $z = 0$ is trivial.

(c) Let $R := \frac{1}{L}$, where $0 < L < \infty$. First assume that $|z| < R$. Then there exists $0 < \delta < \frac{1}{2}$ such that $|z| = R(1 - 2\delta) = \frac{1}{L}(1 - 2\delta)$. Then $\overline{\lim\limits_{k\to\infty}} |a_k|^{\frac{1}{k}} |z| = (1 - 2\delta)$, and therefore $|a_k|^{\frac{1}{k}} < 1 - \delta$ for all sufficiently large k. Then $|a_k| \leq (1 - \delta)^k$ for all large enough k. By the comparison test we obtain that $\sum\limits_{k=0}^{\infty} a_k z^k$ is absolutely convergent. If $|z| > R = \frac{1}{L}$, then $\overline{\lim\limits_{k\to\infty}} |a_k|^{\frac{1}{k}} |z| = L|z| > 1$. Thus, $|a_k z^k| > 1$ for infinitely many values of k. This shows that $\sum\limits_{k=0}^{\infty} a_k z^k$ diverges.

(d) First assume that $R = \infty$. Then $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ converges for all z and

$$\frac{f(z+h)-f(z)}{h} = \sum_{k=0}^{\infty} \frac{a_k[(z+h)^k - z^k]}{h} = \sum_{k=0}^{\infty} k a_k z^{k-1} + \sum_{k=2}^{\infty} a_k b_k \tag{14}$$

where

$b_k = \frac{(z+h)^k - z^k}{h} - kz^{k-1} = \sum\limits_{n=2}^{k} \binom{k}{n} h^{n-1} z^{k-n} \leq |h| \sum\limits_{n=0}^{k} \binom{k}{n} |z|^{k-n} = |h|(|z|+1)^n$ for $|h| \leq 1$.

Therefore, for $|h| \leq 1$, we have the estimate

$$\left|\frac{f(z+h)-f(z)}{h} - \sum_{k=0}^{\infty} k a_k z^{k-1}\right| \leq |h| \sum_{k=0}^{\infty} |a_k|(|z|+1)^k \leq A|h| \text{ for some } A < \infty,$$

since $\sum\limits_{k=0}^{\infty} |a_k| w^k$ converges for all $z+1 = w > 0$. Thus, if $h \to 0$, then $f'(z) = \sum\limits_{k=0}^{\infty} k a_k z^{k-1}$.

Now assume that $0 < R < \infty$. For $|z| < R$ choose $\delta > 0$ so that $|z| = R - 2\delta$. Let $|h| < \delta$. Then $|z+h| \leq |z| + |h| \leq R - 2\delta + \delta = R - \delta < R$ and, as in the previous case, Equation (14) holds with

$$b_n = \sum_{n=2}^{k} \binom{k}{n} h^{n-1} z^{k-n}. \tag{15}$$

If $z = 0$ and $b_k = h^{k-1}$ the proof follows. Otherwise, an estimate for b_k can be found by noting that

$$\binom{k}{n} = \frac{k(k-1)\ldots(k-n+1)}{n!} \leq k^2 \binom{k}{n-2} \text{ for } n \geq 2. \tag{16}$$

Therefore, for $z \neq 0$,

$$|b_k| \leq \frac{k^2|h|}{|z|^2} \sum_{n=2}^{k} \binom{k}{n-2} |h|^{n-2} |z|^{k-(n-2)} \leq \frac{k^2|h|}{|z|^2} \sum_{j=0}^{k} \binom{k}{j} |h|^j |z|^{k-j}$$

$$= \frac{k^2|h|}{|z|^2}(|z|+|h|)^k \leq \frac{k^2|h|}{|z|^2}(R-\delta)^k.$$

and

$$\left|\frac{f(z+h)-f(z)}{h} - \sum_{k=0}^{\infty} k a_k z^{k-1}\right| \leq \frac{|h|}{|z|^2} \sum_{k=0}^{\infty} k^2 |a_k| (R-\delta)^k \leq A|h|,$$

since $z \neq 0$ is fixed and $\sum\limits_{k=0}^{\infty} k^2 |a_k| z^k$ also converges for $|z| < R$. Again, letting $h \to 0$, gives that $f'(z) = \sum\limits_{k=0}^{\infty} k a_k z^{k-1}$. ■

It is obvious that convergent power series $\sum_{k=-\infty}^{\infty} a_k z^k$ with coefficients $a = (a_k)_{k\in\mathbb{Z}} \in m$ form a vector space over $\mathbb{C}$; in particular, if $\sum_{k=-\infty}^{\infty} a_k z^k$ converges for $|z| < R_a = \frac{1}{L_a}$ (where $a \in m$ and $L_a := \overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|}$) and $\sum_{k=-\infty}^{\infty} b_k z^k$ converges for $|z| < R_b = \frac{1}{L_b}$ (where $b \in m$ and $L_b := \overline{\lim_{k\to\infty}} \sqrt[k]{|b_k|}$), then $\sum_{k=-\infty}^{\infty} (a_k + b_k) z^k$ converges for all $|z| \in \mathbb{C}$ with $|z| < \min(R_a, R_b)$. This implies that the radius of convergence R_{a+b} of $\sum_{k=-\infty}^{\infty} (a_k + b_k) z^k$ satisfies

$$R_{a+b} \geq \min(R_a, R_b). \tag{17}$$

Moreover, define $c := a * b$ and let $|z| < \min(R_a, R_b)$. Then $\sum_{k=-\infty}^{\infty} a_k z^k$ exists since

$$\sum_{n=0}^{N} |\sum_{k=0}^{n} a_{n-k} b_k||z|^n \leq \sum_{n=0}^{N} (\sum_{k=0}^{n} |a_{n-k}||b_k|)|z|^n \leq (\sum_{k=0}^{N} |a_n||z|^n)(\sum_{k=0}^{N} |b_n||z|^n). \tag{18}$$

Thus,

$$R_{a*b} \geq \min(R_a, R_b). \tag{19}$$

The immediate consequence of Equations (17) and (19) is the fact that

$$\tilde{m} := \left\{ a \in m \ : \ L_a = \overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|} < \infty \right\}$$

is closed under addition and convolution in the field $(m, +, *)$; i.e., $(\tilde{m}, +, *)$ is a commutative ring with multiplicative identity $I := e_0$. To show that $(\tilde{m}, +, *)$ is, in fact, a subfield of $(m, +, *)$, requires more effort. The Z-transform, introduced in the following section, will allow us to substantiate this claim.

We need the following result from complex analysis. Let D be an open set in $\mathbb{C}$. Recall that a function $f : D \to \mathbb{C}$ is analytic if, for all $z \in D$,

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. The following is one of the main results of complex analysis (for a proof, see [9]). We denote by $U_\epsilon(0)$ a disk in the complex plane centered at the origin with radius $\epsilon > 0$.

Theorem 2.2. *Let $f : U_\epsilon(0) \to \mathbb{C}$. The following are equivalent*

i. *f is analytic.*

ii. *There exists $b_k \in \mathbb{C}$ such that $f(z) = \sum_{k=0}^{\infty} b_k z^k$ for all $z \in U_\epsilon(0)$.*

Moreover, if (ii) holds, then f is infinitely often differentiable on $U_\epsilon(0)$ and $b_k = \frac{f^{(k)}(0)}{k!}$.

■

2.2 The Z-Transform

To define the Z-transform, we first introduce some preliminary definitions. We denote by $U^*_\epsilon(0)$ the punctured disk $U_\epsilon(0)\backslash\{0\}$ and we define

$$\tilde{M} := \left\{ f : \text{ there exists } a \in \tilde{m} \text{ such that } f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \text{ for all } z \in U^*_{R_a}(0) \right\},$$

The next Proposition shows that $\tilde{M}$ is the collection of complex-valued functions which are analytic on some punctured disk $U^*_\epsilon(0)$ and have, at the origin, a pole of finite order.

Proposition 2.3. *$f \in \tilde{M}$ if and only if there exists $k \in \mathbb{Z}$ such that the function $g(z) := z^{-k} f(z)$ satisfies*

(A) *there exists $b \in \tilde{m}$ such that $g(z) = \sum_{k=0}^{\infty} b_k z^k$ for all $z \in U_{R_b}(0)$, and*

(B) *$g(0) = b_0 \neq 0$.*

Proof. By definition, $f \in \tilde{M}$ if and only if there exists $a \in \tilde{m}$ such that

$$f(z) = a_k z^k + a_{k+1} z^{k+1} + a_{k+2} z^{k+2} + \ldots = z^k g(z),$$

where $g(z) = (a_k + a_{k+1} z^1 + a_{k+2} z^2 + \ldots)$ and $g(0) = a_k \neq 0$. ∎

Example 2.4. (a) The function $f : z \to \sin \frac{1}{z}$ is not in $\tilde{M}$. This follows from the fact that, for any $k \in \mathbb{Z}$, $g(z) := z^{-k} f(z) = z^{-k} \sin \frac{1}{z}$ is either undefined at $z = 0$ (for $k \geq 0$), or $g(0) = 0$ (for $k < 0$).
(b) The function $f : z \to \frac{1}{z^2} e^z$ is in $\tilde{M}$ since $g(z) := z^2 f(z) = e^z$ is analytic for all $z \in \mathbb{C}$ and $g(0) = 1 \neq 0$.

∎

Theorem 2.5. *$(\tilde{M}, +, \cdot)$ is a field.*

Proof. The only non-trivial facts to be proven are

(a) that $g_1, g_2 \in \tilde{M}$ if $f_1, f_2 \in \tilde{M}$, and

(b) $\frac{1}{f} \in \tilde{M}$ if $f \in \tilde{M}$.

We show (a) first. Let $f_1(z) = z^{-k_1} g_1(z)$ and $f_2(z) = z^{-k_2} g_2(z)$, where $g_1(z)$, $g_2(z)$ satisfy the properties (A) and (B) from Proposition 2.3. Then $(f_1 \cdot f_2)(z) = z^{-(k_1+k_2)} g_1(z) g_2(z) = z^{-(k_1+k_2)} g(z)$, where

$$g(z) = g_1(z) g_2(z) = (\sum_{k=0}^{\infty} b^1_k z^k)(\sum_{k=0}^{\infty} b^2_k z^k) = \sum_{k=0}^{\infty} (\sum_{i=0}^{k} b^1_{k-i} b^2_i) z^k = \sum_{k=0}^{\infty} c_k z^k.$$

Since $c = b^1 * b^2 \in \tilde{m}$ and $g(0) = c_0 = b_0^1 b_0^2 \neq 0$ it follows that $f_1, f_2 \in \tilde{M}$.

To prove (b), let $f(z) = z^k g(z)$, where g has the properties (A) and (B) from Proposition 2.3. Then $\frac{1}{f(z)} = z^{-k} h(z)$, where $h(z) = \frac{1}{g(z)}$. Since $g(z)$ is analytic on $U_\epsilon(0)$ for some $\epsilon > 0$, and since $g(0) \neq 0$, it follows that there exists $0 < \epsilon_1 \leq \epsilon$ such that $g(z) \neq 0$ for all $z \in U_{\epsilon_1}(0)$. Thus, h is analytic on $U_{\epsilon_1}(0)$ and $h(0) \neq 0$. Since $\sum_{k=0}^{\infty} b_k z^k$ exists for some $z \neq 0$, it follows from Theorem 2.1 that $\overline{\lim_{k\to\infty}} \sqrt[k]{|b_k|} < 0$; i.e., $b \in \tilde{m}$. ∎

Definition 2.6. *The Z-transform, $Z : \tilde{m} \to \tilde{M}$, is defined as*

$$Z(a) := f, \text{ where } f(z) := \sum_{k=-\infty}^{\infty} a_k z^k.$$

∎

Note that $\sum_{k=-\infty}^{\infty} a_k z^k$ exists for all $z \in U_{R_a}(0)$, where $R_a = \frac{1}{L_a}$ and $L_a = \overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|} < \infty$.

Theorem 2.7. *$(\tilde{m}, +, *)$ is a field and the Z-transform, $Z : \tilde{m} \to \tilde{M}$, is a field isomorphism.*

Proof. It is easy to see that $Z : \tilde{m} \to \tilde{M}$ is linear. Moreover, if $a, b \in \tilde{m}$ and $c := a*b$, then it follows from Equation (18) that $\sum_{k=0}^{\infty} c_k z^k$ converges absolutely for $|z| < \min(R_a, R_b)$ (in fact, $\sum_{n=0}^{\infty}(\sum_{k=0}^{\infty} |a_{n-k}||b_k|)|z|^k$ converges). Moreover, $|(\sum_{k=0}^{n} a_k z^k)(\sum_{k=0}^{n} b_k z^k) - \sum_{k=0}^{\infty} c_k z^k|$

$$= |(a_0 + a_1 z + \ldots + a_k z^k)(b_0 + b_1 z + \ldots + b_k z^k) - ((a_0 b_0) + (a_1 b_0 + a_0 b_1) z \ldots)|$$

$$= |a_0 b_0 + (a_1 b_0 + a_0 b_1) z + \ldots + (a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0) z^n + (a_1 b_k + \ldots + a_n b_1) z^{n+1} + \text{(a few terms only)} z^{n+2} - [(a_0 b_0) + (a_1 b_0 + a_0 b_1) z + \ldots]|$$

$$= |\sum_{k=n+1}^{2n} \tilde{c}_k z^k + \sum_{k=2n+1}^{\infty} c_k z^k| \leq \sum_{k=n+1}^{2n} |\tilde{c}_k||z|^k + \sum_{k=2n+1}^{\infty} |c_k||z|^k \leq \sum_{k=n+1}^{\infty} (\sum_{j=0}^{k} |a_{k-j}||b_j|)|z|^k,$$

where $\tilde{c}_k = \sum a_j b_s$ for j or $s \geq n+1$ and $j + s = k$.Thus, if $a, b \in \tilde{m}$, then $c := a * b \in \tilde{m}$ and

$$Z(a) \cdot Z(b) = Z(c) = Z(a * b).$$

This shows that $Z : \tilde{m} \to \tilde{M}$ is linear and multiplicative. Obviously, if $Z(a) = f = 0$, then $f(z) = a_k z^k + a_{k+1} z^{k+1} + \ldots = 0$ for all $z \in U^*_{R_a}(0)$. But then $g(z) := z^{-k} f(z) = a_k + a_{k+1} z + a_{k+2} z^2 + \ldots$ is identical to zero for all $z \in U^*_{R_a}(0)$ and thus for all $z \in U_{R_a}(0)$ (since g is continuous on $z \in U_{R_a}(0)$). But then $g(0) = a_k = 0$, which implies that $a = 0$. Thus, z is one-to-one. By definition, Z is onto. It follows that Z is a linear and multiplicative bijection from $\tilde{m}$ into $\tilde{M}$. Since $\tilde{M}$ is a field, $\tilde{m}$ is a field and the proof is complete. ∎

2.3 Inverses of Finite Sequences

Professor Hsiao-Chun Wu's problem involving finite sequences a with

$$0 = ind_l(a) < ind_u(a) = L < \infty,$$

(see Problem 1.14) can be restated the problem in a slightly different fashion. Recall that

$$\|a^{-1}\|_1 := \sum_{k=0}^{\infty} |a_k^{-1}|$$

and define $\|a^{-1}\|_{1,N} := \sum_{k=0}^{N} |a_k^{-1}|$. Then, given $L, N \in \mathbb{N}$, and $M >> 0$, the problem is to find $a \in m_0^L := \{a \in m \,:\, 0 = ind_l(a) < ind_u(a) = L\}$ such that the mass of a^{-1} is located within the first N coordinates; i.e.,

$$\|a^{-1}\|_{1,N} \geq M(\|a^{-1}\|_1 - \|a^{-1}\|_{1,N}).$$

Since the lower index of a is equal to 0, then we can write

$$a = (\ldots, 0, 0, a_0, a_1, a_2, \ldots, a_L, 0, 0, \ldots),$$

where $n \in \mathbb{N}$ and $a_0 \neq 0$. Without loss of generality, we may assume that $a_L = 1$. The Z-transform of a is then

$$Z(a) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{L-1} z^{L-1} + z^L,$$

which, by the Fundamental Theorem of Algebra, can be written in the form

$$Z(a) = (z - b_1)^{n_1} (z - b_2)^{n_2} \ldots (z - b_j)^{n_j}$$

for some $1 \leq j \leq n$ and some distinct complex numbers $b_k \neq 0$ for $1 \leq k \leq j$, and $n_1 + n_2 + \ldots + n_j = L$. Computing $Z^{-1}(a) = Z(a^{-1})$ and using partial fraction decomposition, we have

$$Z(a^{-1}) = Z^{-1}(a) = \frac{1}{(z-b_1)^{n_1}(z-b_2)^{n_2}\ldots(z-b_j)^{n_j}} = \left(\frac{A_1^1}{(z-b_1)} + \frac{A_2^1}{(z-b_1)^2} \ldots + \frac{A_{n_1}^1}{(z-b_1)^{n_1}}\right) + \ldots$$

$$+\left(\frac{A_1^j}{(z-b_j)} + \frac{A_2^j}{(z-b_j)^2} \ldots + \frac{A_{n_j}^j}{(z-b_j)^{n_j}}\right).$$

From this expansion, we see that it is useful to have a closer look at the inverses of $a = (\ldots, 0, 0, -b, 1, 0, 0, \ldots)$ and its powers (see Lemmas 2.9 and 2.10 below). As a result, we obtain the following regarding the "size" of the inverses of finite sequences.

Theorem 2.8. *Let $a \in m_0^L$ for some $L \geq 1$ such that $a_L = 1$ and*

$$Z(a) = a_0 + a_1 z + \ldots + a_{L-1} z^{L-1} + z^L.$$

Then $\|a^{-1}\|_1 < \infty$ if all roots of $Z(a)$ have absolute value larger than 1.

For the proof of this theorem we need the following two lemmas.

Lemma 2.9. *Let $0 \neq b \in \mathbb{C}$ and $a := (\ldots, 0, 0, -b, 1, 0, 0, \ldots)$. Then*

$$a^{-1} = (\ldots, 0, 0, \underset{\substack{\uparrow \\ 0\text{-th position}}}{\tfrac{-1}{b}}, \tfrac{-1}{b^2}, \tfrac{-1}{b^3}, \ldots).$$

If $0 < |b| \leq 1$, then $\|a^{-1}\|_1 = \infty$. If $|b| > 1$, then $\|a^{-1}\|_1 = \frac{1}{|b|-1}$ and

$$\|a^{-1}\|_{1,N} := \sum_{k=0}^{N} |a_i^{-1}| = (1 - \tfrac{1}{|b|})^{N+1} \|a^{-1}\|_1.$$

Moreover, $\frac{\|a^{-1}\|_{1,N}}{\|a^{-1}\|_1 - \|a^{-1}\|_{1,N}} = |b|^{N+1} - 1$.

Proof. If $a := (\ldots, 0, 0, -b, 1, 0, 0, \ldots)$, then $Z(a) = z - b$ and

$$Z^{-1}(a) = \tfrac{1}{z-b} = -\tfrac{1}{b(1-\frac{z}{b})} = \sum_{j=0}^{\infty} \tfrac{-1}{b^{j+1}} z^j \text{ for } |z| < |b|.$$

Thus $a^{-1} = (\ldots, 0, 0, \frac{-1}{b}, \frac{-1}{b^2}, \frac{-1}{b^3}, \ldots)$. It is clear that $\|a^{-1}\| = \infty$ if $0 < |b| \leq 1$. If $|b| > 1$, then

$$\|a^{-1}\|_1 = \tfrac{1}{|b|} \sum_{j=0}^{\infty} \tfrac{1}{|b|^j} = \tfrac{1}{|b|} \tfrac{1}{1-\frac{1}{|b|}} = \tfrac{1}{|b|-1}.$$

Now let $\|a^{-1}\|_{1,N} := \sum_{k=0}^{N} |a_i^{-1}|$. Then, setting $x := \frac{1}{|b|}$,

$$\sum_{k=0}^{N} |a_i^{-1}| = x + x^2 + \ldots + x^{N+1} = x(1 + x + \ldots + x^N) = x\tfrac{1-x^{N+1}}{1-x}$$

$$= \tfrac{1}{|b|} \tfrac{1-(\frac{1}{|b|})^{N+1}}{1-\frac{1}{|b|}} = \tfrac{1-(\frac{1}{|b|})^{N+1}}{|b|-1} = (1 - (\tfrac{1}{|b|})^{N+1}) \left\|a_b^{-1}\right\|_1.$$

With this result, we have

$$\|a^{-1}\|_1 - \|a^{-1}\|_{1,N} = \tfrac{1}{|b|-1} - \tfrac{1-(\frac{1}{|b|})^{N+1}}{|b|-1} = \tfrac{(\frac{1}{|b|})^{N+1}}{|b|-1},$$

and

$$\tfrac{\|a^{-1}\|_{1,N}}{\|a^{-1}\|_1 - \|a^{-1}\|_{1,N}} = \tfrac{1-(\frac{1}{|b|})^{N+1}}{|\frac{1}{b}|}^{N+1} = |b|^{N+1} - 1.$$

$\blacksquare$

Lemma 2.10. *Let $a_b := (\ldots, 0, 0, -b, 1, 0, 0, \ldots)$ for some $0 \neq b \in \mathbb{C}$ with $|b| > 1$ and $a := a_b^n = a_b * a_b * \ldots * a_b$ for some $n \in \mathbb{N}$. Then*

$$\left\|a_b^{-n}\right\|_1 = \left\|a_b^{-1}\right\|_1^n = \tfrac{1}{(|b|-1)^n}.$$

Proof. Since $Z(a) = (Z - b)^n$, then it follows that

$$Z(a^{-1}) = Z^{-1}(a) = \frac{1}{(z-b)^n} = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\frac{1}{z-b} = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\sum_{k=0}^{\infty}\frac{-1}{b^{k+1}}z^k$$

$$= \frac{(-1)^n}{(n-1)!}\sum_{k=0}^{\infty}\frac{1}{b^{k+1}}k(k-1)\dots(k-(n-2))z^{k-(n-1)}$$

$$= \frac{(-1)^n}{(n-1)!}\sum_{k=n-1}^{\infty}\frac{1}{b^{k+1}}\frac{k!}{(k-(n-1))!}z^{(k-(n-1))} = (-1)^n\sum_{k=n-1}^{\infty}\frac{1}{b^{k+1}}\binom{k}{n-1}z^{(k-(n-1))}$$

$$= (-1)^n\sum_{j=0}^{\infty}\frac{1}{b^{j+n}}\binom{j+n-1}{n-1}z^j = \frac{(-1)^n}{b^n}\sum_{j=0}^{\infty}\frac{1}{b^j}\binom{j+n-1}{n-1}z^j$$

and

$$a^{-1} = a_b^{-1} = (\dots, 0, 0, \tfrac{(-1)^n}{b^n}, \tfrac{(-1)^n}{b^{n+1}}n, \tfrac{(-1)^n}{b^{n+2}}\tfrac{n(n-1)}{2}, \dots, a_j^{-1}, \dots),$$

where $\frac{(-1)^n}{b^n}$ is in the 0 position and $a_j^{-1} = \frac{(-1)^n}{b^{n+j}}\binom{j+n-1}{n-1}$ for $j \geq 0$. Since $|b| > 1$, we use $x = \frac{1}{|b|}$ to obtain

$$\|a^{-1}\|_1 = \frac{1}{|b|^n}\sum_{j=0}^{\infty}\frac{1}{|b|^j}\binom{j+n-1}{n-1} = x^n\sum_{j=0}^{\infty}x^j\binom{j+n-1}{n-1} = \sum_{j=0}^{\infty}x^{n+j}\frac{(j+n-1)!}{(n-1)!j!}.$$

Letting $k = j + n - 1$, or equivalently $j = k - (n-1)$, then

$$\sum_{j=0}^{\infty}x^{n+j}\frac{(j+n-1)!}{(n-1)!j!} = \frac{1}{(n-1)!}\sum_{k=n-1}^{\infty}x^{k+1}\frac{k!}{(k-(n-1))!}$$

$$= \frac{1}{(n-1)!}\sum_{k=n-1}^{\infty}(k)(k-1)\dots(k-(n-2))x^{k+1}$$

$$= \frac{1}{(n-1)!}\sum_{k=0}^{\infty}(k)(k-1)\dots(k-(n-2))x^{k+1}$$

$$= \frac{1}{(n-1)!}x^n\sum_{k=0}^{\infty}(k)(k-1)\dots(k-(n-2))x^{k-(n-1)}$$

$$= \frac{1}{(n-1)!}x^n\frac{d^{n-1}}{dx^{n-1}}\sum_{k=0}^{\infty}x^k = \frac{1}{(n-1)!}x^n\frac{d^{n-1}}{dx^{n-1}}\frac{1}{1-x} = \frac{1}{(n-1)!}x^n\frac{(n-1)!}{(1-x)^n}$$

$$= \frac{x^n}{(1-x)^n} = (\tfrac{1}{|b|})^n\frac{1}{(1-\frac{1}{|b|})^n} = \frac{1}{(|b|-1)^n}.$$

This shows that, in particular $\left\|a_b^{-n}\right\|_1 = \frac{1}{(|b|-1)^n}$. Now consider

$$a_b^{-1} = (\dots, 0, 0, -\tfrac{1}{b}, -\tfrac{1}{b^2}, -\tfrac{1}{b^3}, \dots).$$

where $ind_l(a_b^{-1}) = 0$. Then

$$\left\|a_b^{-1}\right\|_1^n = (\sum_{j=1}^{\infty}(\tfrac{1}{|b|})^j)^n = (\tfrac{1}{|b|}\sum_{j=0}^{\infty}(\tfrac{1}{|b|})^j)^n = (\tfrac{1}{|b|}\tfrac{1}{(1-\frac{1}{|b|})})^n = \tfrac{1}{(|b|-1)^n}$$

■

Proof. Now for the proof of the Theorem 2.8. Let $a \in m_0^L$ for some $L \geq 1$ such that

$$Z(a) = a_0 + a_1 z + \ldots + a_{L-1}z^{L-1} + z^L \text{ with } a_0 \neq 0.$$

Then there exists j distinct complex numbers $b_j \neq 0$, with $1 \leq j \leq n$ such that

$$Z(a) = (z-b_1)^{n_1}(z-b_2)^{n_2}\ldots(z-b_j)^{n_j},$$

where $n_1 + n_2 + \ldots + n_j = L$. Therefore, if $a_b := (\ldots, 0, 0, b, -1, 0, 0, \ldots)$, then

$$Z(a) = Z(a_{b_1})^{n_1} Z(a_{b_2})^{n_2} \ldots Z(a_{b_j})^{n_j} = Z(a_{b_1}^{n_1} * a_{b_2}^{n_2} * \ldots * a_{b_j}^{n_j}),$$

or $a = a_{b_1}^{n_1} * a_{b_2}^{n_2} * \ldots * a_{b_j}^{n_j}$. In particular, if $|b_k| > 1$ for all $1 \leq k \leq j$, then $a^{-1} = a_{b_1}^{-n_1} * a_{b_2}^{-n_2} * \ldots * a_{b_j}^{-n_j}$ and by Lemma 2.10,

$$\|a^{-1}\|_1 = \left\|a_{b_1}^{-n_1} * a_{b_2}^{-n_2} * \ldots * a_{b_j}^{-n_j}\right\|_1 \leq \left\|a_{b_1}^{-n_1}\right\|_1 \cdot \left\|a_{b_2}^{-n_2}\right\|_1 \cdot \ldots \cdot \left\|a_{b_j}^{-n_j}\right\|_1$$

$$= \left\|a_{b_1}^{-1}\right\|_1^{n_1} \cdot \left\|a_{b_2}^{-1}\right\|_1^{n_2} \cdot \ldots \cdot \left\|a_{b_j}^{-1}\right\|_1^{n_j} = \tfrac{1}{(|b_1|-1)^{n_1}} \cdot \tfrac{1}{(|b_2|-1)^{n_2}} \cdot \ldots \cdot \tfrac{1}{(|b_j|-1)^{n_j}}.$$ ■

Corollary 2.11. *Let $a \in m_0^L$ such that all the roots of $Z(a) = a_0 + a_1 z + \ldots + a_{L-1}z^{L-1} + a_L z^L$ have absolute value greater than some $q > 1$. Then*

$$\|a^{-1}\|_1 \leq \tfrac{1}{|a_L|}\tfrac{1}{(q-1)^L}.$$

Proof. Let $\tilde{a} := \frac{1}{a_L}a$. Then $a^{-1} = \frac{1}{a_L}\tilde{a}^{-1}$. By Theorem 2.8, then

$$\|\tilde{a}^{-1}\|_1 \leq \tfrac{1}{(|b_1|-1)^{n_1}} \cdot \tfrac{1}{(|b_2|-1)^{n_2}} \cdot \ldots \cdot \tfrac{1}{(|b_j|-1)^{n_j}} \leq \tfrac{1}{(q-1)^{n_1}} \cdot \tfrac{1}{(q-1)^{n_2}} \cdot \ldots \cdot \tfrac{1}{(q-1)^{n_j}}$$

$$= \tfrac{1}{(q-1)^L}. \text{ (since } n_1 + n_2 + \ldots + n_j = L)$$ ■

Next we characterize those infinite sequences a which have a finite inverse a^{-1}; i.e., $ind_u(a) = \infty$ and $ind_u(a^{-1}) < \infty$.

Proposition 2.12. *Let $a \in m$ with $ind_u(a) < \infty$. Then $ind_u(a^{-1}) < \infty$ if and only if $Z(a)$ is a rational function of the form $\frac{z^k}{p(z)}$ for some $k \in \mathbb{Z}$.*

Proof. Let $ind_u(a^{-1}) < \infty$. Then

$$Z(a^{-1}) = a_j z^j + a_{j+1}z^{j+1} + \ldots + a^{j+k}z^{j+k} = z^j(a_j + a_{j+1}z + \ldots + a_{j+k}z^k) = z^j p(z)$$

for some $j \in \mathbb{N}$. So we have $Z(a) = Z(a^{-1})^{-1} = \frac{1}{z^j p(z)} = \frac{z^k}{p(z)}$ with $k = -j$. If $Z(a) = \frac{z^k}{p(z)}$, then

$$Z(a^{-1}) = Z(a)^{-1} = \tfrac{p(z)}{z^k} = b_0 z^{-k} + b_1 z^{-k+1} + \ldots + b_n z^{n-k},$$

and thus a^{-1} is finite. ■

We end this section with an alternative way to approach the question put forward by Professor Wu (see Problem 1.14). Let s be an unknown signal given as a sequence $r = h*s$ in a finite filter $h = (h_0, h_1, h_2, \ldots, h_L, 0, 0, \ldots)$. To reconstruct the original sequence s from the received signal r we have to compute $s = h^{-1} * r$. Since h^{-1} is infinite, "cut" the sequence h after N terms in order to be able to implement it. That is, look for $s_N = h_N^{-1} * r$, where $h_N^{-1} = (h_0^{-1}, h_1^{-1}, h_2^{-1}, \ldots, h_N^{-1}, 0, 0, \ldots)$. Now the question becomes how the cut-off affects the distance between the true signal s and the reconstructed signal s_N; that is, we are interested in finding

$$\|s - s_N\| = \left\|r * h^{-1} - r * h_N^{-1}\right\| \leq \|r\| \left\|h^{-1} - h_N^{-1}\right\| \leq \|s\| \, \|h\| \left\|h^{-1} - h_N^{-1}\right\|$$

For example, taking the one norm, $\|\cdot\|_1$, the main question becomes estimating

$$\left\|h^{-1} - h_N^{-1}\right\|_1 = \sum_{k=N+1}^{\infty} |h_k^{-1}|.$$

If we take $h = a_b = (\ldots, 0, 0, -b, 1, 0, 0, \ldots)$ for $|b| > 1$ (see Lemma 2.9), then

$$h^{-1} = (\ldots, 0, 0, \tfrac{-1}{b}, \tfrac{-1}{b^2}, \tfrac{-1}{b^3}), \ldots)$$

and therefore,

$$\left\|h^{-1} - h_N^{-1}\right\|_1 = \sum_{k=N+1}^{\infty} \tfrac{1}{|b|^{k+1}} = \tfrac{1}{|b|^{N+2}} \sum_{k=0}^{\infty} \tfrac{1}{|b|^k} = \tfrac{1}{|b|^{N+2}} \tfrac{1}{1-\frac{1}{|b|}}$$

$$= \tfrac{1}{|b|^{N+1}} \tfrac{1}{|b|-1} = \tfrac{1}{|b|^{N+1}} \left\|h^{-1}\right\|_1.$$

This implies that

$$\|s - s_N\|_1 \leq \|s\|_1 \|h\|_1 \tfrac{1}{|b|^{N+1}} \left\|h^{-1}\right\|_1 = \|s\|_1 \tfrac{|b|+1}{|b|-1} \tfrac{1}{|b|^{N+1}}.$$

Thus, if one has an initial rough estimate of the size $\|s\|_1$ of the true signal s, then one can choose either b large and N small or $|b|$ close to 1 and N large to reconstruct s to a given degree of precision.

2.4 Fibonacci Sequences

Sequences labelled as Fibonacci were studied in 1202 by Leonardo Pisano (nicknamed Fibonacci). Fibonacci wrote a number of texts which played an important role in reviving ancient mathematical skills and he made significant contributions of his own. Fibonacci lived in the days before printing, so his books were hand written and the only way to have a copy of one of his books was to have another hand-written copy made. Of his books we still have copies of *Liber abaci* (1202), which is based on the arithmetic and algebra that Fibonacci had accumulated during his travels. *Liber abaci*, which went on to be widely copied and imitated, introduced the Hindu-Arabic place-valued decimal system and the use of Arabic numerals into Europe. Although mainly a book about the use of Arab numerals, simultaneous linear equations are also studied in this work. A problem in the third section of *Liber abaci* led to the introduction of the Fibonacci numbers and the Fibonacci sequence for which Fibonacci is best remembered today (see Figure 1):

> A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

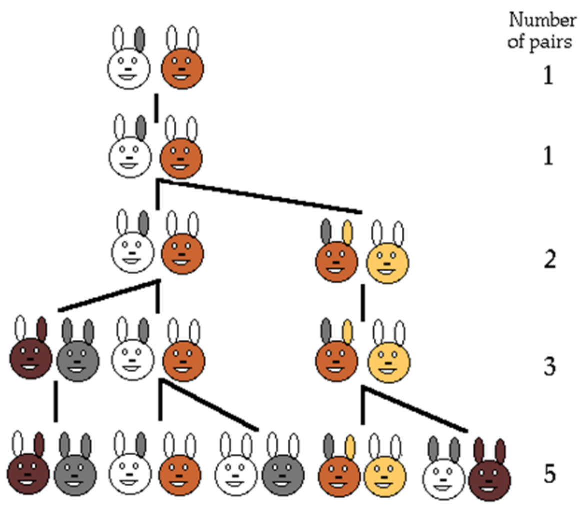

Figure 1: Rabbit Problem

The resulting sequence is $a = (\ldots, 0, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots)$, which has $ind_l(a) = 1$. The sequence a is such that each number is the sum of the two preceding numbers. The computation of the first few terms in the sequence is straightforward, but the direct computation of the number of rabbits for month k, where k is large, is cumbersome, if not impossible. The Z-transform can be used to find the number in the sequence for any month k (see also [13] and [11]).

Theorem 2.13 (Generalized Fibonnacci Sequence)**.** *Let* $a = (\ldots, 0, 0, a_0, a_1, a_2, \ldots)$, $ind_l(a) = 0$ *where* a_0 *and* a_1 *are given and*

$$a_{k+2} = Aa_{k+1} + Ba_k \tag{20}$$

with $A, B \in \mathbb{C}$ *and* $k > 0$*. Then the closed form for* a_k *is given by*

$$a_k = \frac{-c}{z_0^{k+1}} - \frac{d}{z_1^{k+1}} \tag{21}$$

where $z_0 = \frac{-A+\sqrt{A^2+4B}}{2B}$, $z_1 = \frac{-A-\sqrt{A^2+4B}}{2B}$, $c = \frac{z_0(Aa_0-a_1)-a_0}{B(z_0-z_1)}$ *and* $d = \frac{z_1(Aa_0-a_1)-a_0}{B(z_1-z_0)}$.

Proof. If there exists $a \in \tilde{m}$ that solves Equation (20), then the Z-transform of a is given by $Z(a) = f(z) = a_0 + a_1 z + a_2 z^2 + \ldots = \sum_{k=0}^{\infty} a_k z^k$. Then, with the recurrence relation $a_{k+2} = Aa_k + Ba_{k-1}$,

$$f(z) = a_0 + a_1 z + \sum_{k=2}^{\infty}(Aa_{k-1} + Ba_{k-2})z^k = a_0 + a_1 z + \sum_{k=2}^{\infty} Aa_{k-1}z^k + \sum_{k=2}^{\infty} Ba_{k-2}z^k$$

$$= a_0 + a_1 z + z\sum_{k=1}^{\infty} Aa_k z^k + z^2 \sum_{k=0}^{\infty} Ba_k z^k$$

and

$$\tfrac{f(z-a_0-a_1 z)}{z^2} = \tfrac{A}{z}\sum_{k=1}^{\infty} a_k z^k + B\sum_{k=0}^{\infty} a_k z^k = \tfrac{A}{z}(\sum_{k=0}^{\infty} a_k z^k - a_0) + Bf(z) =$$

$$= \tfrac{A}{z}(f(z) - a_0) + Bf(z).$$

Solving for $f(z)$ and using partial fractions, then

$$f(z) = \tfrac{z(Aa_0-a_1)-a_0}{Bz^2+Az-1} = \tfrac{c}{z-z_0} + \tfrac{d}{z-z_1},$$

where $z_0 = \frac{-A+\sqrt{A^2+4B}}{2B}$ and $z_1 = \frac{-A-\sqrt{A^2+4B}}{2B}$. Solving for c and d, then $c = \frac{z_0(Aa_0-a_1)-a_0}{B(z_0-z_1)}$ and $d = \frac{z_1(Aa_0-a_1)-a_0}{B(z_1-z_0)}$.

Now use the Geometric Series $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ to write $f(z)$ in summation form; i.e.,

$$f(z) = \tfrac{c}{z-z_0} + \tfrac{d}{z-z_1} = \tfrac{-c}{z_0(1-\frac{z}{z_0})} - \tfrac{d}{z_1(1-\frac{z}{z_1})} = \tfrac{-c}{z_0}\sum_{k=1}^{\infty}\tfrac{z^k}{z_0^k} - \tfrac{d}{z_1}\sum_{k=1}^{\infty}\tfrac{z^k}{z_1^k}$$

$$= \sum_{k=0}^{\infty}(\tfrac{-c}{z_0^{k+1}} - \tfrac{d}{z_1^{k+1}})z^k.$$

Since Z is one-to-one, we get that $a_k = \frac{-c}{z_0^{k+1}} - \frac{d}{z_1^{k+1}}$. It is easy to see that $a = (a_k)_{k\geq 0} \in \tilde{m}$. ■

From Theorem 2.13 we see that the problem of finding the sequence value at n is made easier by transformation of the sequence into a power series where we can use standard techniques. The multiplying rabbit scenario can be solved by the following corollary.

Corollary 2.14. *Let* $a = (\ldots, 0, 0, a_0, a_1, a_2, \ldots)$, *where* $a_0 = a_1 = 1$ *are given and* $a_{k+2} = a_{k+1} + a_k$ *for* $k \geq 0$. *Then*

$$a_k = \tfrac{1}{\sqrt{5}}[(\tfrac{1+\sqrt{5}}{2})^{k+1} - (\tfrac{1-\sqrt{5}}{2})^{k+1})].$$

Proof. From Theorem 2.13, we have $A = B = a_0 = a_1 = 1$ and we can compute that

$$z_0 = \tfrac{-1+\sqrt{1^2+4(1)}}{2(1)} = \tfrac{-1+\sqrt{5}}{2},\ z_1 = \tfrac{-1-\sqrt{1^2+4(1)}}{2(1)} = \tfrac{-1-\sqrt{5}}{2},$$

$$c = \tfrac{z_0(Aa_0-a_1)-a_0}{z_0-z_1} = \tfrac{-1}{z_0-z_1} = \tfrac{-1}{\sqrt{5}} \text{ and } d = \tfrac{z_1(Aa_0-a_1)-a_0}{z_1-z_0} = \tfrac{-1}{z_1-z_0} = \tfrac{-1}{\sqrt{5}}$$

From Equation (21), then

$$a_k = \frac{-(\frac{-1}{\sqrt{5}})}{(\frac{-1+\sqrt{5}}{2})^{k+1}} - \frac{-(\frac{-1}{\sqrt{5}})}{(\frac{-1-\sqrt{5}}{2})^{k+1}} = \frac{1}{\sqrt{5}}\frac{2^{k+1}}{(-1+\sqrt{5})^{k+1}} - \frac{1}{\sqrt{5}}\frac{2^{k+1}}{(-1-\sqrt{5})^{k+1}}$$

$$= \tfrac{1}{\sqrt{5}}[(\tfrac{2}{-1+\sqrt{5}})^{k+1} - (\tfrac{2}{-1-\sqrt{5}})^{k+1})].$$

Using the fact that $\frac{2}{-1+\sqrt{5}} = \frac{1+\sqrt{5}}{2}$ and $\frac{2}{-1-\sqrt{5}} = \frac{1-\sqrt{5}}{2}$, then

$$a_k = \tfrac{1}{\sqrt{5}}[(\tfrac{1+\sqrt{5}}{2})^{k+1} - (\tfrac{1-\sqrt{5}}{2})^{k+1})].$$

■

In Corollary 2.14, the second term in the expression for a_k is negligible, as k grows large, therefore a_k can be approximated by $\tilde{a}_k$, where

$$\tilde{a}_k = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{k+1}. \tag{22}$$

Corollary 2.14 also reveals the necessity of death in the bunny scenario, as the amount of time it takes to fill the universe with bunnies is less that the average life span of a human being. The current estimate for the size of the universe is 1.9 x 10^{23} cubic light years, where each cubic light year is 4.05 x 10^{107} cubic meters. Thus, the size of the universe is 7.695 x 10^{130} cubic meters. Assuming that each pair of bunnies requires one cubic meter of space, then one could fit 1.539 x 10^{131} bunnies in the universe. From Equation (22), the value of k for which $\tilde{a}_k$ equals 1.539 x 10^{131} is found by solving the following equation,

$$1.539 \text{ x } 10^{131} = \tfrac{1}{\sqrt{5}}(\tfrac{1+\sqrt{5}}{2})^{k+1}$$

Using logarithms, then

$$\ln(1.539 \text{ x } 10^{131}) = \ln(\tfrac{1}{\sqrt{5}}) + (k+1)\ln\tfrac{1+\sqrt{5}}{2}, \quad \text{and}$$

$$k = \frac{\ln(1.539 \text{ X } 10^{131}) - \ln(\frac{1}{\sqrt{5}})}{\ln(\frac{1+\sqrt{5}}{2})} - 1 = 628.4 \text{ months.}$$

Therefore, if the bunny life-span is more than 53 years, it would take approximately 52.3 years to fill the universe with bunnies!

Corollary 2.15. *Let a be the sequence $a = (\ldots, 0, 0, a_0, a_1, a_2, \ldots)$ where $a_0 = 0$, $a_1 = 1$, and $a_{k+2} = a_{k+1} - a_k$ for $k \geq 0$. Then*

$$a_k = \tfrac{1}{6}[(3+i\sqrt{3})(\tfrac{1+i\sqrt{3}}{2})^{k+1} + (-3+i\sqrt{3})(\tfrac{1-i\sqrt{3}}{2})^{k+1}].$$

Proof. From Theorem 2.13, we have $A = 1$, $B = -1$ and we can compute that

$$z_0 = \frac{-1+\sqrt{1^2-4(1)}}{2(-1)} = \frac{1-i\sqrt{3}}{2}, \quad z_1 = \frac{-1-\sqrt{1^2-4(1)}}{2(-1)} = \frac{1+i\sqrt{3}}{2},$$

$$c = \frac{z_0(Aa_0-a_1)-a_0}{z_0-z_1} = \frac{\frac{1-i\sqrt{3}}{2}(-1)}{z_0-z_1} = \frac{i\sqrt{3}-1}{2(-i\sqrt{3})} = \frac{-3-i\sqrt{3}}{6}, \text{ and}$$

$$d = \frac{z_1(Aa_0-a_1)-a_0}{z_1-z_0} = \frac{\frac{1+i\sqrt{3}}{2}(-1)}{z_1-z_0} = \frac{-1-i\sqrt{3}}{2i\sqrt{3}} = \frac{-3+i\sqrt{3}}{6}.$$

From Equation (21), then

$$a_k = \frac{-c}{z_0^{k+1}} + \frac{d}{z_1^{k+1}} = \frac{\frac{3+i\sqrt{3}}{6}}{(\frac{1-i\sqrt{3}}{2})^{k+1}} + \frac{\frac{-3+i\sqrt{3}}{6}}{(\frac{1+i\sqrt{3}}{2})^{k+1}} = \frac{2^k}{3}\left(\frac{3+i\sqrt{3}}{(1-i\sqrt{3})^{k+1}} + \frac{-3+i\sqrt{3}}{(1+i\sqrt{3})^{k+1}}\right).$$

$$= \tfrac{1}{6}[(3+i\sqrt{3})(\tfrac{2}{1-i\sqrt{3}})^{k+1} + (-3+i\sqrt{3})(\tfrac{2}{1+i\sqrt{3}})^{k+1}]$$

$$= \tfrac{1}{6}[(3+i\sqrt{3})(\tfrac{1+i\sqrt{3}}{2})^{k+1} + (-3+i\sqrt{3})(\tfrac{1-i\sqrt{3}}{2})^{k+1}].$$

■

From the recurrence relation, we compute the sequence a in Corollary 2.15 as

$$a = (\ldots, \underset{\substack{\uparrow \\ \text{0-th position}}}{0}, 0, 1, 1, 0, 0, -1, -1, 0, 0, 1, 1, 0, 0, -1, -1, 0, \ldots), \tag{23}$$

and note that the closed form of a_k contains complex numbers and irrational numbers whereas the sequence in Equation (23) does not. Corollaries 2.14 and 2.15 illustrate a suprising and powerful relationship between the integers and the irrational/complex numbers.

3 THE ASYMPTOTIC Z-TRANSFORM

In this chapter we discuss an extension of the Z-transform from the field $(\tilde{m}, +, *)$ to the field $(m, +, \cdot)$. Since $Z : \tilde{m} \to \tilde{M}$ is an isometric isomorphism, one can expect that the extension Z_{as} maps m into a larger space M_{as}. To define M_{as} we start with the following definition.

Definition 3.1. Consider a complex valued function f defined on a sectorial region

$$S = \{z \ : \ 0 < |z| < R, \ |\arg z| \leq \theta\}$$

where $\theta \geq 0$ and $R > 0$. Then $a \in m$ is said to represent $f(z)$ asymptotically as $z \to 0$ if for all $k \geq ind_l(a)$,

$$\frac{f(z) - \sum_{i=-\infty}^{k} a_i z^i}{z^k} \to 0 \text{ as } z \to 0.$$

The notation $f \approx a$ is typically used to denote this asymptotic representation. Note that $f \approx 0$ $(0 \in m)$ at 0 $(0 \in \mathbb{C})$ if for $z \in S$,

$$\lim_{z \to 0} \frac{f(z)}{z^k} = 0 \text{ for all } k \in \mathbb{Z}.$$

$\blacksquare$

A typical example of a function that is asymptotically equal to zero (at $0 \in \mathbb{C}$) is given by

$$f(z) = e^{\frac{-1}{z}}$$

on $S = \{z \neq 0, \ |\arg z| < \theta\}$, for $\theta < \frac{\pi}{2}$. Indeed, if $z \in S$, then $z = |z|e^{i\alpha}$ for some $|\alpha| \leq \theta$. Thus $|f(z)| = |e^{\frac{-1}{|z|}e^{-i\alpha}}| = e^{\frac{-1}{|z|}\cos\alpha} \leq e^{\frac{-1}{|z|}\cos\theta}$. Since $\cos\theta > 0$ it follows that for $z \in S$,

$$\lim_{z \to 0} \frac{f(z)}{z^k} = 0$$

for all $k \geq 0$, and thus for all $k \in \mathbb{Z}$ (it is obvious that the statement holds for $k < 0$). We define

$$[0] := \{f \ : \ f \approx 0\}.$$

If f is a complex-valued function defined on some sectorial region S, then $[f] := f + [0]$ and

$$M_{as} := \{[f], \ f \approx a \text{ on a sectorial region } S \text{ for some } a \in m\}.$$

Proposition 3.2. *Let f be a complex valued function on some sectorial region S. The following statements are equivalent*

i. *There exists $a \in m$ such that $f \approx a$.*

ii. *There exists* $k_0 \in \mathbb{Z}$ *such that*

(a) *for* $z \in S$, $\lim\limits_{z\to 0} \frac{f(z)}{z^{k_0}} = a_{k_0} \neq 0$, *and*

(b) $\lim\limits_{z\to 0} \dfrac{f(z) - \sum\limits_{i=-\infty}^{k-1} a_i z^i}{z^k} = a_k$ *for all* $k > k_0$.

Proof. Suppose that statement (i) holds; that is,

$$\frac{f(z) - \sum\limits_{i=-\infty}^{k} a_i z^i}{z^k} \to 0$$

as $z \to 0$ for all $k \geq ind_l(a) =: k_0$. If $k = k_0$, then $\sum\limits_{i=-\infty}^{k} a_i z^i = a_{k_0} z^{k_0}$ and therefore $\frac{f(z) - a_{k_0} z^{k_0}}{z_{k_0}} \to 0$ or $\frac{f(z)}{z^{k_0}} \to a_{k_0} \neq 0$ for $z \to 0$. If $k > k_0$, then

$$\sum_{i=-\infty}^{k} a_i z^i = \sum_{i=-\infty}^{k-1} a_i z^i + a_k z^k$$

and therefore

$$\frac{f(z) - \sum\limits_{i=-\infty}^{k-1} a_i z^i - a_k z^k}{z^k} \to 0 \quad \text{or} \quad \frac{f(z) - \sum\limits_{i=-\infty}^{k-1} a_i z^i}{z^k} \to a_k$$

as $z \to 0$. This shows that (i) implies (ii). If (ii) holds, then define

$$a = (\ldots, 0, 0, a_{k_0}, a_{k_0+1}, a_{k_0+2}, \ldots).$$

Then, as above, one shows that

$$\frac{f(z) - \sum\limits_{i=-\infty}^{k} a_i z^i}{z^k} \to 0$$

for all $k \geq k_0 = ind_l(a)$. Thus $f \approx a$; that is, (i) holds. ■

Example 3.3. Let $S := (0, \infty)$ and define $f(z) = \sin\left(\frac{1}{z}\right)$ for $z > 0$. Then $f \notin M_{as}$. Indeed, $f(\frac{1}{n\pi}) = 0$ for all $n \in \mathbb{N}$ and therefore it is impossible that there exists k_0 such that $\lim\limits_{0 \leftarrow z \in S} \frac{f(z)}{z^{k_0}} = a_{k_0} \neq 0$.

■

Proposition 3.4. (a) *If* $[f] \cap [g] \neq \emptyset$, *then* $[f] = [g]$.

(b) $[f \cdot g] = [f][g]$.

(c) $[f] + [g] = [f + g]$.

Proof. (a) Let $h \in [f] \cap [g]$. Then there exist $o_i \in [0]$ such that $h = f + o_i$ and $h = g + o_2$. But then $0 = f - g + o_1 - o_2$ and thus $g = f + o_3$, where $o_3 = o_1 - o_2 \in [0]$. But then $[g] = g + [0] = f + o_3 + [0] = f + [0] = [f]$.
(b) Let $f_1 \in [f]$ where $f \approx a$ and $g_1 \in [g]$ where $g \approx b$. Then $f_1 = f + o_1$ and $f_2 = g + o_2$. Thus, $f_1 \cdot f_2 = f \cdot g + f \cdot o_1 + g \cdot o_2 + o_1 \cdot o_2$. Since $f \approx a$, we have that

$$\tfrac{f(z)}{z^{k_0}} \to a_{k_0} \neq 0 \text{ for } k_0 = ind_l(a).$$

Thus

$$\tfrac{f(z)\cdot o_1(z)}{z^k} = \tfrac{f(z)}{z^{k_0}} \tfrac{o_1(z)}{z^{k-k_0}} \to 0$$

as $z \to 0$ for all $k \in \mathbb{Z}$. This shows that $f \cdot o_1 \in [0]$. Similarly, $g \cdot o_2 \in [0]$ and $o_1 \cdot o_2 \in [0]$. Thus, $f_1 \cdot f_2 = f \cdot g + o_3$ where $o_3 = f \cdot o_1 + g \cdot o_2 + o_1 \cdot o_2 \in [0]$. Therefore, $f_1 \cdot f_2 \in [f \cdot g]$ or $[f][g] = [fg]$. If $h \in [fg]$ then $h = fg + o$ for some $o \in 0$. Thus $h \in [f][g] + [0] = [f] \cdot [g]$ or $[fg] \subset [f][g]$. This shows that $[fg] = [f][g]$. The proof of statement (c) is straight forward. ∎

Proposition 3.5. *Let $a \in \tilde{m}$ and $f(z) := \sum\limits_{i=-\infty}^{\infty} a_i z^i$. Then $f \approx a$.*

Proof. The statement follows from the fact that

$$\frac{f(z) - \sum\limits_{i=-\infty}^{k} a_i z^i}{z^k} = \frac{\sum\limits_{i=k+1}^{\infty} a_i z^i}{z^k} = a_{k+1} z + a_{k+2} z^2 + a_{k+3} z^3 + \ldots = z g(z) \to 0$$

where $g(z) = a_{k+1} + a_{k+2} z + a_{k+3} z^2 + \ldots$ is analytic on $U_{R_a}(0)$. ∎

Lemma 3.6. *Let $a \in \tilde{m}$ and $j \in \mathbb{Z}$. Then $f \approx a$ if and only if $z^j f(z) \approx e_j * a$.*

Proof. Since $e_j * a = (a_{i-j})_{i \in \mathbb{Z}}$, the statement follows from the fact that

$$\frac{z^j f(z) - \sum\limits_{i=-\infty}^{k} a_{i-j} z^i}{z^k} = \frac{f(z) - \sum\limits_{i=-\infty}^{k} a_{i-j} z^{i-j}}{z^{k-j}} = \frac{f(z) - \sum\limits_{i=-\infty}^{k-j} a_i z^i}{z^{k-j}}.$$

∎

The next theorem is known as Ritt's Theorem and was first published in 1916 (see also [10] and [12]).

Theorem 3.7 (Ritt's Theorem)**.** *Let $a \in m$. Then there exists f such that $f \approx a$.*

Proof. Let $a \in m$ with $k_a = ind_l(a)$. Then by Lemma 3.6, there exists f such that $f \approx a$ if and only if there exists $\tilde{f}$ (where $\tilde{f}(z) = z^{-k_a} f(z)$) such that $\tilde{f} \approx \tilde{a} := e_{-k_a} * a$. Thus, we may assume without restriction of generality that

$$a = (\ldots, 0, 0, a_0, a_1, a_2, \ldots).$$

Now let $a \in m$, $a = (\ldots, 0, 0, a_0, a_1, a_2, \ldots)$, where a is not necessarily in $\tilde{m}$. We will construct a function $f(z)$ such that $f \approx a$. The idea is to replace the potentially divergent series $\sum_{k=0}^{\infty} a_k z^k$ by a convergent series of the form

$$f(z) = \sum_{k=0}^{\infty} a_k \alpha_k(z) z^k, \tag{24}$$

where the convergence factors $\alpha_k(z)$ are chosen in such a way that the new series is uniformly convergent but shares the asymptotic properties at 0 of the original. To do so, the function $\alpha_k(z)$ must be small for $z \neq 0$ (to ensure convergence) and it must rapidly tend to one as $z \to 0$ (to ensure that the behavior of f at 0 is determined solely by the sequence a). One type of function which can be adapted so as to have the desired properties is

$$\alpha_k(z) := 1 - e^{(\frac{-b_k}{z^\beta})} \text{ with } b_k > 0. \tag{25}$$

The exponent β is some number in the interval $0 < \beta < 1$. With β sufficiently small, we can be sure that the exponent has negative real part in any given sector S of the plane that does not contain $(0, \infty]$. The following fact is needed for the proof:

$$|1 - e^z| = |z \int_0^1 e^{tz} \text{ dt}| \leq |z| \text{ for } \Re z \leq 0. \tag{26}$$

Equations (25) and (26) then $|a_k \alpha_k(z) z^k| \leq |a_k||b_k||z|^{k-\beta}$. Define $b_k = |a_k|^{-1}$ for $a_k \neq 0$ and $b_k = 0$ for $a_k = 0$, then the following inequality holds

$$f(z) := \sum_{k=1}^{\infty} a_k \alpha_k(z) z^k \leq \sum_{k=1}^{\infty} |z|^{k-\beta}$$

which converges uniformly for $|z| \leq z_0 < 1$. Now we show that the function f satisfies $f \approx a$. From Equations (24) and (25),

$$z^{-m}(f(z) - \sum_{k=0}^{m} a_k z^k) = -\sum_{k=0}^{m} a_k \exp\left(\tfrac{-b_k}{z^\beta}\right) z^{-(m-k)} + \sum_{k=m+1}^{\infty} a_k \alpha_k(z) z^{k-m}$$

The first term on the right hand side tends to zero as $z \to 0$, and for $|z| \leq z_0$,

$$|\sum_{k=m+1}^{\infty} a_k \alpha_k(z) z^{k-m}| \leq \sum_{k=m+1}^{\infty} |z|^{k-m-\beta} < \tfrac{|z|^{1-\beta}}{1-|z|}.$$

The middle term approaches zero with z, and we have the desired result. ■

The proof remains valid for $z_0 \geq 1$ if $b_k = |a_k z_0^k|^{-1}$.

Proposition 3.8. *If $f \approx a$ and $g \approx b$ for some $a, b \in m$, then $f \cdot g \approx a * b$.*

Proof. Let $k_a = ind_l(a)$ and $k_b = ind_l(b)$. Then

$$\tilde{f}(z) := z^{-k_a} f(z) \approx e_{-k_a} * a = (\ldots, 0, 0, \tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \ldots) = \tilde{a}, \text{ and}$$

$$\tilde{g}(z) := z^{-k_b} g(z) \approx e_{-k_b} * b = (\ldots, 0, 0, \tilde{b}_0, \tilde{b}_1, \tilde{b}_2, \ldots) = \tilde{b}.$$

where $\tilde{a}_i = a_{k_a+i}$ and $\tilde{b}_i = b_{k_b+i}$. Define

$$\tilde{f}_k(z) := \frac{\tilde{f}(z) - \sum_{i=-\infty}^{k} \tilde{a}_i z^i}{z^k} \quad \text{and} \quad \tilde{g}_k(z) := \frac{\tilde{g}(z) - \sum_{i=-\infty}^{k} \tilde{b}_i z^i}{z^k}.$$

It follows from Lemma 3.6 that $\tilde{f}_k(z) \to 0$ and $\tilde{g}_k(z) \to 0$ as $z \to 0$. Moreover, since $\tilde{a}_i = \tilde{b}_i = 0$ for $i < 0$, we have that

$$\tilde{f}(z) = \sum_{i=0}^{k} \tilde{a}_i z^i + z^k \tilde{f}_k(z) \quad \text{and} \quad \tilde{g}(z) = \sum_{i=0}^{k} \tilde{b}_i z^i + z^k \tilde{g}_k(z).$$

Thus,

$$\begin{aligned}
\tilde{f}(z)\tilde{g}(z) &= (\sum_{i=0}^{k} \tilde{a}_i z^i)(\sum_{i=0}^{k} \tilde{b}_i z^i) + (\sum_{i=0}^{k} \tilde{a}_i z^i) z^k \tilde{g}_k(z) + (\sum_{i=0}^{k} \tilde{b}_i z^i) z^k \tilde{f}_k(z) + z^{2k} \tilde{f}_k(z)\tilde{g}_k(z) \\
&= \sum_{i=0}^{k} c_i z^i + \sum_{i=k+1}^{2k} \tilde{c}_i z^i + (\sum_{i=0}^{k} \tilde{a}_i z^i) z^k \tilde{g}_k(z) + (\sum_{i=0}^{k} \tilde{b}_i z^i) z^k \tilde{f}_k(z) + z^{2k} \tilde{f}_k(z)\tilde{g}_k(z),
\end{aligned}$$

where $c_i = \sum_{j+s=i} \tilde{a}_j \tilde{b}_j = (\tilde{a} * \tilde{b})_i$ and $\tilde{c}_i = \sum_{j+s=i} \tilde{a}_j \tilde{b}_s$, for $0 \le j \le k$ and $0 \le s \le k$. Thus,

$$\frac{\tilde{f}(z)\tilde{g}(z) - \sum_{i=0}^{k} c_i z^i}{z^k} = \sum_{i=k+1}^{2k} \tilde{c}_i z^{i-k} + (\sum_{i=0}^{k} \tilde{b}_i z^i) \tilde{f}_k(z) + (\sum_{i=0}^{k} \tilde{a}_i z^i) \tilde{g}_k(z) + z^k \tilde{f}_k(z)\tilde{g}_k(z).$$

It follows that if $k \ge 0 = ind_l(\tilde{a} * \tilde{b})$, then

$$\frac{\tilde{f}(z)\tilde{g}(z) - \sum_{i=0}^{k} (\tilde{a} * \tilde{b}) z^i}{z^k} \to 0 \text{ as } z \to 0.$$

This shows that $\tilde{f}\tilde{g} \approx \tilde{a} * \tilde{b}$ or, equivalently, that $z^{-(k_a+k_b)} fg \approx e_{-(k_a+k_b)} * a * b$. Then by Lemma 3.6, $fg \approx a * b$. ■

We are now in the position to define the asymptotic Z-transform $Z_{as} : m \to M_{as}$ as

$$Z_{as}(a) = \widehat{a} = [f],$$

where $f \approx a$. First note that Z_{as} is well defined. Indeed, by Theorem 3.7, there exists f such that $f \approx a$. Moreover, if $f_1 \approx a$ and $f_2 \approx a$, then $f_1 - f_2 \approx 0$, and therefore $[f_1] = [f_2]$.

Theorem 3.9. *$(M_{as}, +, \cdot)$ is a field and $Z_{as} : m \to M_{as}$ is a field isomorphism.*

Proof. If $Z_{as}(a) = [f]$ and $Z_{as}(b) = [g]$, then $f \approx a$ and $g \approx b$. Thus, by Proposition 3.8, $fg \approx a * b$. Therefore, $Z_{as}(a)Z_{as}(b) = [f][g] = [fg] = Z_{as}(a * b)$. This shows that $Z_{as} : m \to M_{as}$ is multiplicative (clearly Z_{as} is also additive and linear). By the definition of M_{as}, Z_{as} is onto. It remains to be shown that Z_{as} is one-to-one. If $Z_{as}(a) = [0]$, suppose that $a \neq 0$. Then

$$\frac{0 - \sum_{i=-\infty}^{k} a_i z^i}{z^k} \to 0 \text{ for all } k \geq ind_l(a).$$

Let $k_0 = ind_l(a)$. Then $a_{k_0} \neq 0$ and

$$\frac{0 - \sum_{i=-\infty}^{k_0} a_i z^i}{z^{k_0}} = a_{k_0} \to 0,$$

which is a contradiction. This shows that Z_{as} is an isomorphism. Since m is a field, it follows that M_{as} is also a field. ■

Example 3.10. A familiar example of an asymptotic series appears in the problem of calculating the "exponential integral" [3],

$$Ei(x) = \int_{-\infty}^{x} e^t \tfrac{1}{t} \text{ dt, for } x < 0.$$

Successive integration by parts shows that

$$Ei(x) = e^x \tfrac{1}{x}[1 + \tfrac{1}{x} + \tfrac{2!}{x^2} + \ldots + \tfrac{m!}{x^m} + R_m(x)],$$

where $R_m(x) = (m+1)!x \int_{-\infty}^{x} e^{t-x} \frac{1}{t^{m+2}}$ dt. One more integration by parts yields

$$\begin{aligned} |R_m(x)| &= |(m+1)!x[[e^{t-x}\tfrac{1}{t^{m+2}}]_{t=-\infty}^{t=x} + \int_{-\infty}^{x} e^{t-x}\tfrac{m+2}{t^{m+3}} \text{ dt}]| \\ &\leq (m+1)!\tfrac{1}{|x|^{m+1}} + (m+2)!|x| \int_{-\infty}^{x} \tfrac{1}{t^{m+3}} \text{ dt} \\ &= (m+1)!\tfrac{1}{|x|^{m+1}} + (m+1)!\tfrac{1}{|x|^{m+1}} = 2(m+1)!\tfrac{1}{|x|^{m+1}}. \end{aligned}$$

Substituting $z = \frac{-1}{x}$ and defining

$$f(z) := -e^{\frac{1}{z}}\tfrac{1}{z}Ei(\tfrac{-1}{z}) = \tfrac{-1}{z}e^{\frac{1}{z}} \int_{-\infty}^{\frac{-1}{z}} e^t \tfrac{1}{t} \text{ dt}$$

for $z > 0$, we obtain that

$$|\tfrac{f(z)-(1-z+2!z^2-\ldots+(-1)^k k! z^k)}{z^k}| = |\tfrac{1}{z^k}R_k(\tfrac{-1}{z})| \leq 2(k+1)!|z| \to 0$$

as $z \to 0$. Thus, if $a := (\ldots, 0, 0, 1, -1, 2!, -3!, \ldots, (-1)^m m!, \ldots)$, then $Z_{as}(a) = [f]$. Observe that, in this example, $a \notin \tilde{m}$, so $Z(a)$ is not well defined.

$\blacksquare$

Example 3.11. Consider the recurrence relation $a_{n+1} = Aa_n + b_n$, $(n \geq 0)$ with a_0 given and $b = (b_n)_{n\in\mathbb{N}} \in m$ given. Let

$$a = (a_0, a_1, a_2, \ldots),\ b = (b_0, b_1, b_2, \ldots), \text{ and}$$

$$c = (a_1, a_2, a_3, \ldots) = e_{-1} * (a - a_0 e_0).$$

We wish to find a such that $c = e_{-1} * (a - a_0 e_0) = Aa + b$. Then

$$e_{-1} * a - Aa = a_0 e_{-1} + b \Leftrightarrow \widehat{e}_{-1}\widehat{a} - A\widehat{a} = a_0 \widehat{e}_{-1} + \widehat{b}$$

$$\Leftrightarrow \tfrac{1}{z}\widehat{a}(z) - A\widehat{a}(z) = \tfrac{a_0}{z} + \widehat{b}(z) \Leftrightarrow (\tfrac{1}{z} - A)\widehat{a}(z) = \tfrac{a_0}{z} + \widehat{b}(z)$$

$$\Leftrightarrow \widehat{a}(z) = \tfrac{a_0}{1-zA} + \tfrac{z}{I-zA}\widehat{b}(z).$$

Since $\frac{a_0}{1-zA} = \sum_{i=0}^{\infty} a_0 A^i z^i$ and $\widehat{e}_1(z) = z$, we have

$$\widehat{a}(z) = \widehat{d}(z) + \widehat{e}_1(z)\widehat{d}(z)\widehat{b}(z),$$

where $d = (\ldots, 0, 0, a_0, a_0 A, a_0 A^2, \ldots)$. Thus $a = d + e_1 * d * b$.

$\blacksquare$

Notice, that if $b \notin \tilde{m}$, then the solution method requires the use of the asymptotic version Z_{as} of the classical Z-tranform. Without the use of Z_{as} (or an equivalent procedure in the field $(m, +, *)$ as illustrated in the next example), this problem could not have been solved in a mathematically rigorous way.

Example 3.12. Consider the convolution equation

$$a_{n+1} = \sum_{i=0}^{n} d_{n-i} a_i + b_n = (d * a)_n + b_n, \tag{27}$$

where $a_0 \in \mathbb{C}$, $b, d \in m$ are given. Let

$$a = (a_0, a_1, a_2, \ldots),\ b = (b_0, b_1, b_2, \ldots),\ d = (d_0, d_1, d_2, \ldots), \text{ and}$$

$$\tilde{a} = (a_1, a_2, a_3, \ldots) = e_{-1} * (a - a_0 e_0).$$

Then Equation (27) is equivalent to

$$\tilde{a} = d * a + b. \tag{28}$$

To find a, we find that Equation (28) holds if and only if

$$e_{-1} * a - a_0 e_{-1} = d * a + b \Leftrightarrow (e_{-1} - d) * a = a_0 e_{-1} + b \Leftrightarrow a = (e_{-1} - d)^{-1} * (a_0 e_{-1} + b).$$

To find $(e_{-1} - d)^{-1}$, observe that $e_{-1} - d = e_{-1} * (e_0 - e_1 * d)$. Therefore, $(e_{-1} - d)^{-1} = e_1 * (I - e_1 * d)^{-1}$. To be able to compute $(I - e_1 * d)^{-1}$, let us assume that d is a finite sequence. Then $e_1 * d = (\ldots, 0, 0, d_0, d_1, d_2, \ldots)$ and $I - e_1 d = (\ldots, 0, 0, 1, -d_0, -d_1, \ldots)$, where d_0 and $-d_0$ are respectively in the first position. Then

$$\begin{aligned}
\widehat{I - e_1 d} &= 1 - d_0 z - d_1 z^2 - \ldots - d_N z^{N+1} \\
&= -d_N [\tfrac{-1}{d_N} + \tfrac{d_0}{d_N} z + \ldots + \tfrac{d_{N-1}}{d_N} z^N + z^{N+1}] \\
&= -d_N \prod_{j=1}^{k} (z - b_j)^{n_j},
\end{aligned}$$

for some pairwise distinct complex numbers b_j. Thus

$$(\widehat{I - e_1 d})^{-1} = \frac{\frac{-1}{d_N}}{\prod_{j=1}^{k} (z - b_j)^{n_j}} \text{ and } (I - e_1 d)^{-1} = \tfrac{-1}{d_N} \prod_{j=1}^{k} a_{b_j}^{-n_j},$$

where $a_{b_j} = (\ldots, 0, 0, -b_j, 1, 0, 0, \ldots)$ as in Lemma 2.9.

■

If d is not finite, we do not (yet) have an effective method to compute $(I - e_1 d)^{-1}$.

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APPENDIX: COMPUTING THE CONVOLUTION INVERSE WITH MATHEMATICA

The following Mathematica program will compute the convolution inverse, a^{-1}, given $a \in m$. The input data is the lower and upper indices of a and the first 100 nonzero elements of a. The output is the lower index of a^{-1} and the first 100 elements of a^{-1}. The program can be suitably modified for sequences a with $ind_u(a) - ind_l(a) > 100$.

```
(Label[beginofprogram])
indexlower=Input["Enter lower index of sequence:"]
indexupper=Input["Enter the upper index of sequence:"]
a=Table[i*0,i,1,100]
Do[a[[i]]=Input["Enter next(first) number in the sequence:"],i,1,indexupper-indexlower+1]
Print["Starting position of sequence: ",indexlower]
Print[a]
ainvindexlower=-indexlower
Print["Starting position of inverse: ",ainvindexlower]
ainv=Table[i*0,i,1,100]
ainv[[1]]=1/a[[1]]
m=1
(Label[beginloop1];
    m=m+1;
    ainv[[m]]=0;
    i=0;
    Label[beginloop2];
        i=i+1;
        ainv[[m]]=ainv[[m]]+a[[1-i+m]]*ainv[[1+i-1]];
        If[i < m - 1,Goto[beginloop2]]);
    ainv[[m]]=-ainv[[m]]/a[[1]];
    If[m < 100,Goto[beginloop1]])
Print[ainv]
(Label[endofprogram])
```

VITA

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